Bernuau spline wavelets and Sturmian sequences

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Abstract

We present spline wavelets of class $C^n(\mathbb{R})$ supported by sequences of aperiodic discretizations of $\mathbb{R}$. The construction is based on multiresolution analysis recently elaborated by G. Bernuau. At a given scale, we consider discretizations that are sets of left-hand ends of tiles in a self-similar tiling of the real line with finite local complexity. Corresponding tilings are determined by two-letter Sturmian substitution sequences. We illustrate the construction with examples having quadratic Pisot-Vijayaraghavan units (like $\tau = (1+\sqrt{5})/2$ or $\tau^2 = (3+\sqrt{5})/2$) as scaling factor. In particular, we present a comprehensive analysis of the Fibonacci chain and give the analytic form of related scaling functions and wavelets. We also give some hints for the construction of multidimensional spline wavelets based on stone-inflation tilings in arbitrary dimension.

1 Introduction

Under the name "wavelet" is commonly understood a function $\psi(x) \in L^2(\mathbb{R})$ such that the family of functions $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ for $j, k \in \mathbb{Z}$ forms an orthonormal (or at least a Riesz) basis for $L^2(\mathbb{R})$. A function generating through dilatations and translations an orthonormal basis for $L^2(\mathbb{R})$ can be found through a "multiresolution analysis of $L^2(\mathbb{R})$" (shortly MRA), a method settled by S. Mallat [1]. The dilatation factor is usually $\theta = 2$. Indeed, the construction of a wavelet basis within the MRA framework relies on the fact that the lattices $2^{-j}\mathbb{Z}$ are increasing for the inclusion. This property is preserved only when θ is an integer. Then one can raise a natural question: what about choosing another number θ as a scaling factor? A first answer is given in the work of P. Auscher [2]. The following problem was considered: given a real number $\theta > 1$, does there exist a finite set $\{\psi_1, \psi_2, \dots, \psi_\ell\}$ of functions in $L^2(\mathbb{R})$ such that the family $\theta^{j/2}\psi_i(\theta^j x - k),\ j, k \in \mathbb{Z},\ 1 \leq i \leq \ell$, is an orthonormal basis for $L^2(\mathbb{R})$? The author then proved that a basis of this type exists if θ is a rational number. More precisely, for $\theta = p/q > 1$, p and q being relatively prime integers, there exists a set of $p - q$ wavelet functions satisfying the previous condition. It was still an open question whether there exists another generalization of wavelet basis with an irrational scaling factor. On the other hand, in 1992 Buhmann and Micchelli [3] proposed a construction of a wavelet spline basis corresponding to non-uniform and non-self-similar knot sequences. Further [4, 5, 6] studies have been recently devoted to this problem in higher dimension, like answering the question of characterizing functions ψ, dilation sets $\mathcal{D}$, and translation sets $\mathcal{T}$, such that $\left\{|\det(D)|^{\frac{1}{2}}\psi(Dx - \lambda) \,|\, D \in \mathcal{D}, \lambda \in \mathcal{T}\right\}$ forms an orthonormal basis for $L^2(\mathbb{R})$ [6]. As a matter of fact, it was proved in [5] that for any real expansive (*i.e.* all eigenvalues have modulus greater than one) $d \times d$ matrix A, there exists a measurable $E \subset \mathbb{R}^d$ such that $\left\{|\det A|^{j/2}\bar{\hat{\chi}}_E(A^j x - l),\ j \in \mathbb{Z}, l \in \mathbb{Z}^d\right\}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$.

In 1996, a Haar wavelet basis with an algebraic irrational scaling factor and only one generating wavelet $\psi(x)$ was given in [7]. This wavelet basis of $L^2(\mathbb{R})$ lives on the nested sets $\tau^j\mathbb{Z}_\tau$ having the following structure: $\tau^{j/2}\psi(\tau^j x - \lambda),\ j \in \mathbb{Z},\ \lambda \in \Lambda$, where $\tau = (1 + \sqrt{5})/2$ is the golden mean. Set $\mathbb{Z}_\tau$ means set of "tau-integers" and set Λ is the set of admissible translations. One outcome of the latter work is that admissible translations are generically irrational for an irrational scaling factor. This seems to be a common feature

to aperiodic translational sets. In 1998 G. Bernuau [8, 9] settled a general construction of spline wavelets living on locally finite (more precisely with finite local complexity) and self-similar Delaunay (or Delone) sets. His approach was mainly inspired by important results previously obtained by P.G. Lemarié-Rieusset [10] in the more general framework of stratified nilpotent Lie groups. We learn from these last works that irrational factors combined with specific properties of sequence of discretizations imply finitely many generating wavelets and that we also need more functions named "scaling functions" in an appropriate modification of MRA. Actually the number of spline scaling functions and spline wavelets does not depend only on the scaling factor but also on the polynomial order of these functions.

In a previous letter [11], we have presented a definition of multiresolution analysis for an infinite sequence $\cdots \subset \mathcal{F}/\tau^{2j-2} \subset \mathcal{F}/\tau^{2j} \subset \mathcal{F}/\tau^{2j+2} \subset \cdots$ of aperiodic discretizations of $\mathbb{R}$. Corresponding wavelets have been defined and the elementary Haar example was given as an illustration of the method. Starting from the Haar case, one could be attempted to explore less trivial examples in which compact support and orthogonality are required for wavelet families of class C^n, $n \geq 0$. This is a strong constraint for such *Daubechies-like* wavelets which live on aperiodic discretizations, and their existence is not guaranteed at the moment. Our present strategy is to drop out the orthogonality condition and to rather explore certain well-known quasiperiodic counterparts of the dyadic spline wavelets. In the present paper, we shall closely follow the procedure rigorously settled by G. Bernuau. Let us briefly summarize the general setting of the Bernuau construction. Let $\Lambda \subset \mathbb{R}$ be a Delaunay point set in the real line. By Delaunay we mean that Λ is *uniformly discrete* (the distances between any pair of points in Λ are greater than a fixed $r > 0$) and *relatively dense* (there exists $R > 0$ such that $\mathbb{R}$ is covered by intervals of length $2R$ centered at points of Λ). In addition to this Delaunay structure, we demand that the set Λ be

1. *self-similar*: there exists a number $\theta > 1$ (*inflation factor*) such that

$$\theta\Lambda \subset \Lambda, \tag{1}$$

2. *with finite local complexity*: for all $R > 0$, the point set

$$\bigcup_{\lambda\in\Lambda} \{(\Lambda - \lambda) \cap (-R, R)\} \tag{2}$$

is finite. This means that local environments of points in Λ are not different in infinite fashions.

Typically, such sets Λ are mathematical models for one-dimensional structures having a long-range order, like quasicrystals. Our aim here is to construct a Riesz basis of $L^2(\mathbb{R})$, the elements of which are of the affine wavelet type:

$$\theta^{j/2}\psi_\kappa(\theta^j x-\kappa),\ j\in\mathbb{Z},\ \kappa\in\theta^{-1}\Lambda,\ \mathrm{succ}(\kappa)\notin\Lambda, \tag{3}$$

where $\mathrm{succ}(\kappa)$ is the nearest right neighbour of κ in the set $\theta^{-1}\Lambda$, and for which the set $\{\psi_\kappa(x)\}$ of *mother wavelets* is finite and the simplest possible.

At this point, we recall that (v_n) is a Riesz basis of a separable Hilbert space V if and only if each $v\in V$ can be expressed uniquely as $v=\sum_n a_n v_n$ and there exist positive constants A and $B, 0<A\le B$, such that

$$A\sum_n |a_n|^2 \le \Big\|\sum_n a_n v_n\Big\|^2 \le B\sum_n |a_n|^2 \tag{4}$$

for all sequence of scalars a_n. We can say that the v_n's are strongly linearly independent.

In the next section, we shall present a survey of preliminary results concerning the space of splines based on Λ and at the heart of the construction of wavelets and corresponding multiresolution analysis. In Section 3, we recall the Bernuau theorems about the existence and the characterization of the wavelet basis itself. Section 4 is devoted to the description of Delaunay sets based on two-letter Sturmian substitution sequences. These sets have as scaling factor special algebraic integers, with generic symbol β, and named *quadratic Pisot-Vijayaraghavan units* or more simply UPV_2. Among the latter one finds those numbers which are of interest in quasicrystalline studies: $\tau=(1+\sqrt{5})/2$, $\tau^2=(3+\sqrt{5})/2$ (for pentagonal and decagonal cases), $\omega=1+\sqrt{2}$, $\omega^2=3+2\sqrt{2}$ (for octogonal case), and $\delta=2+\sqrt{3}$ (for dodecagonal case). In this context, we shall consider two types of point sets: the β-integers $\mathbb{Z}_\beta$ and certain *model sets* included in the latter. Original results are given in Sections 5–7. Haar wavelets living on β-integer sets with corresponding scaling equations are given in Section 5. Section 6 is devoted to the study of one important example of model set, namely the Fibonacci chain, from a lexicographical point of view (see Proposition 6 and Properties 7-9) and prepares the section 7 in which related splines and wavelets together with their scaling equations are explicitly constructed. In Section 8, we consider a class of self-similar tilings of $\mathbb{R}^d$ having the so-called stone-inflation

property. For instance, we can find stone-inflation tilings among Penrose or triangle tilings of the plane. We just sketch the method of construction of Bernuau spline wavelets adapted to such tilings.

2 Space of splines for multiresolution analysis

2.1 The definition of the space $V_0^{(s)}(\Lambda)$

Any Delaunay set Λ determines a space of splines of order s, $s \geq 2$, in the following way

Definition 1. *Let $s \geq 2$. Then $V_0^{(s)}(\Lambda)$ is the closed subspace of $L^2(\mathbb{R})$ defined by* $V_0^{(s)}(\Lambda) = \left\{ f(x) \in L^2(\mathbb{R}) \,\middle|\, \dfrac{\mathrm{d}^s}{\mathrm{d}x^s} f(x) = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \right\}$.

An equivalent definition is given in terms of the restriction of functions to intervals determined by consecutive elements of Λ. Suppose the latter is defined by the increasing one-to-one map from $\mathbb{Z}$: $\mathbb{Z} \ni n \to \lambda_n$, $\cdots \lambda_{n-1} < \lambda_n < \lambda_{n+1} \cdots$. There results from Def. 1 that $V_0^{(s)}(\Lambda) = \Big\{ f \in C^{s-2},\ f \in L^2(\mathbb{R}),\ f_{|[\lambda_n, \lambda_{n+1}]}$ is a polynomial of degree $\leq s-1 \Big\}$. Therefore, $V_0^{(s)}(\Lambda)$ is the space of splines of order s with nodes in Λ. Let us now give a classical result about the existence of a Riesz basis for $V_0^{(s)}(\Lambda)$ [12].

Theorem 1. *For all Delaunay sets $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ and for all $s \geq 2$, there exists a Riesz basis $\{B_\lambda^{(s)}, \lambda \in \Lambda\}$ of $V_0^{(s)}(\Lambda)$. The function $B_\lambda^{(s)}$ (called B-spline) is the unique function in $V_0^{(s)}(\Lambda)$ satisfying the following conditions:*

(i) $\operatorname{supp} B_\lambda^{(s)} = [\lambda, \lambda']$, *where $\lambda' \in \Lambda$.*

(ii) The interval (λ, λ') contains exactly $s-1$ points of Λ.

(iii) $\displaystyle\int_{\mathbb{R}} B_\lambda^{(s)} = \frac{\lambda' - \lambda}{s}$.

See [12] for proof. Note that (i) and (ii) give precise information on the (compact) support of $B_\lambda^{(s)}$ whilst (iii) is a normalization condition.

2.2 Construction of B-splines of order s, $B_\lambda^{(s)}(x)$

The construction of $B_\lambda^{(s)}(x)$ can be carried out in various ways:

- by recurrence [13], with the use of the formula valid for all $\lambda_n \in \Lambda$

$$B_{\lambda_n}^{(1)}(x) = \begin{cases} 1 & \text{if } \lambda_n \leq x < \lambda_{n+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$B_{\lambda_n}^{(s)}(x) = \omega_{s,n}(x) B_{\lambda_n}^{(s-1)}(x) + (1 - \omega_{s,n+1}(x))\, B_{\lambda_{n+1}}^{(s-1)}(x), \text{ for } s \geq 2 \tag{6}$$

  with $\omega_{s,n}(x) = (x - \lambda_n)/(\lambda_{n+s-1} - \lambda_n)$,

- by using the condition of minimal support, which means that $\forall f \in V_0^{(s)}(\Lambda)$,

$$\text{supp } f \subset \text{supp } B_\lambda^{(s)} \Rightarrow f \propto B_\lambda^{(s)}, \tag{7}$$

- by inverse Fourier transform which gives the explicit form

$$B_{\lambda_n}^{(s)}(x) \propto \text{Fourier}^{-1} \left[\frac{1}{(i\xi)^s} \left(\sum_{l=0}^{s} a_l(n) e^{-i\lambda_{n+l}\xi} \right) \right], \tag{8}$$

  in which $\{a_l(n)\}$ is the unique solution of the linear system

$$\begin{aligned} &\sum_{l=0}^{s} (\lambda_{n+l})^j x_l = 0, \quad 0 \leq j \leq s-1, \\ &\sum_{l=0}^{s} (\lambda_{n+l})^s x_l = \frac{(-1)^s}{s!}. \end{aligned} \tag{9}$$

Note that the above linear system is easily solved since it involves Vandermonde determinants:

$$a_l(n) = \frac{(-1)^l}{s!} \left[\prod_{0 \leq l' < l} (\lambda_{n+l} - \lambda_{n+l'}) \prod_{l < l' \leq s} (\lambda_{n+l'} - \lambda_{n+l}) \right]^{-1}. \tag{10}$$

From (8) one can check that the s^{th} derivative of $B_{\lambda_n}^{(s)}(x)$ is given as a finite linear superposition of Dirac masses located at the points λ_{n+l}, $0 \leq l \leq s$:

$$\frac{\mathrm{d}^s}{\mathrm{d}x^s} B_{\lambda_n}^{(s)}(x) \propto \sum_{l=0}^{s} a_l(n) \delta_{\lambda_{n+l}}.$$

Since the Fourier transform of $B^{(s)}_{\lambda_n}(x+\lambda_n)$ depends on the s-tuple $(\lambda_{n+1}-\lambda_n,\ldots,\lambda_{n+s}-\lambda_n)$ only and since such s-tuples assume their values in a finite set for n varying in $\mathbb{Z}$ if Λ has finite local complexity, we can assert the following:

Proposition 1. *Let $\Lambda\subset\mathbb{R}$ be a Delaunay set of finite local complexity. Then the B-splines of order s based on Λ are of the form $B^{(s)}_\lambda(x)=\phi_\lambda(x-\lambda),\ \lambda\in\Lambda$, where the set $\{\phi_\lambda(x),\ \lambda\in\Lambda\}$ is a finite set of functions with compact support.*

Therefore, in the finite local complexity case, it is possible to partition the indexing set $\mathbb{Z}$ for Λ into a finite set of equivalence classes $\bar n$, $\mathbb{Z}=\bigcup_{\bar n=\bar 0}^{\bar q}\bar n$ where class $\bar n$ is defined by

$$\bar n=\left\{k\in\mathbb{Z}\ \mid\ B^{(s)}_{\lambda_k}(x+\lambda_k)=B^{(s)}_{\lambda_n}(x+\lambda_n)\ \forall x\right\}.$$

Correspondingly, for a given s, the point set Λ is partitioned into $\Lambda=\bigcup_{\bar n=\bar 0}^{\bar q}\Lambda_{\bar n}$ with $\Lambda_{\bar n}=\{\lambda_k\in\Lambda\,|\,k\in\bar n\}$. The relation $\lambda_k\in\Lambda_{\bar n}$ means that λ_k and λ_n are left-hand ends of identical s-letter words if we identify each interval $(\lambda_k,\lambda_{k+1})$ with a letter of the allowed alphabet. To each class $\bar n$ is biunivocally associated the function $\phi_{\bar n}(x)\equiv\phi_{\lambda_k}(x)=B^{(s)}_{\lambda_k}(x+\lambda_k),\ k\in\bar n$. In this way, the space $V^{(s)}_0(\Lambda)$ decomposes into the direct sum

$$V^{(s)}_0(\Lambda)=\bigoplus_{\bar n=\bar 0}^{\bar q}V_{0,\bar n},\tag{11}$$

where $V_{0,\bar n}$ is the closure of the linear span of the functions $\phi_{\bar n}(x-\lambda_k),\ k\in\bar n$.

2.3 Self-similarity and multiresolution analysis

Let Λ be a Delaunay set of finite local complexity and self-similar with inflation factor $\theta>1$ $(\theta\Lambda\subset\Lambda)$. Changing the scale allows us to define subspaces $V^{(s)}_j(\Lambda),\ j\in\mathbb{Z}$.

Definition 2. $V^{(s)}_j(\Lambda)=\left\{f(x)\in L^2(\mathbb{R})\ \mid\ \frac{\mathrm{d}^s}{\mathrm{d}x^s}f(x)=\sum_{\lambda\in\Lambda}a_\lambda\delta_{\theta^{-j}\lambda}\right\}.$

We now have at our disposal an inductive chain of spaces allowing analysis at any scale. More precisely, with the above notations,

Proposition 2. *The sequence of subspaces* $(V_j^{(s)}(\Lambda))_{j\in\mathbb{Z}}$ *is a* θ*-multiresolution analysis of* $L^2(\mathbb{R})$*, i.e.*

(i) for any $j \in \mathbb{Z}$*,* $V_j^{(s)}(\Lambda)$ *is a closed subspace of* $L^2(\mathbb{R})$*,*

(ii) $\cdots \subset V_{-1}^{(s)}(\Lambda) \subset V_0^{(s)}(\Lambda) \subset V_1^{(s)}(\Lambda) \subset \cdots$*,*

(iii) $\bigcup_{j\in\mathbb{Z}} V_j^{(s)}(\Lambda)$ *is dense in* $L^2(\mathbb{R})$*,*

(iv) $\bigcap_{j\in\mathbb{Z}} V_j^{(s)}(\Lambda) = \{0\}$*,*

(v) $f(x) \in V_j^{(s)}(\Lambda)$ *if and only if* $f(\theta^{-j}x) \in V_0^{(s)}(\Lambda)$*,*

(vi) there exists a finite number of functions $\phi_{\bar{n}}(x) \in V_0^{(s)}$*, called* <u>*scaling functions*</u> *such that* $\{\phi_{\bar{n}}(x-\lambda_k)\}_{k\in\bar{n},\,\bar{0}\le\bar{n}\le\bar{q}}$ *is a Riesz basis in* $V_0^{(s)}$*.*

The proof is straightforward from Definitions and Proposition 1.

3 Spline wavelet basis

As is well known, the whole wavelet basis is obtained by scaling the elements of a wavelet subfamily living at a given scale of the multiresolution. For convenience, we choose here the 0^{th}-scale and this subfamily is precisely a basis of the orthogonal complement $W_{-1}^{(s)}(\Lambda)$ of $V_{-1}^{(s)}(\Lambda)$ in $V_0^{(s)}(\Lambda)$:

$$V_0^{(s)}(\Lambda) = V_{-1}^{(s)}(\Lambda) \oplus_\perp W_{-1}^{(s)}(\Lambda). \tag{12}$$

This decomposition can be interpreted as the building up of the "$(j=0)^{\text{th}}$-scale" content of the multiresolution (*i.e.* $V_0^{(s)}$) from the adding of necessary "details" (*i.e.* $W_{-1}^{(s)}$) to the existing content at the next larger scale (*i.e.* $V_{-1}^{(s)}$). A crucial step in the characterization of these "details" is the following existence theorem.

Theorem 2. *(Bernuau [8, 9] Let* $\Lambda \in \mathbb{R}$ *be a Delaunay set of finite local complexity, self-similar with scaling factor* $\theta > 1$*. Then, for all* $s \ge 2$*, there exists a Riesz basis of* $W_{-1}^{(s)}(\Lambda)$ *of the form*

$$\left\{\zeta_{\lambda_n}^{(s)}(x-\lambda_n),\ \lambda_n \in \Lambda,\ \lambda_{n+1} \notin \theta\Lambda\right\}, \quad \text{where } \left\{\zeta_{\lambda_n}^{(s)}(x)\right\}$$

is a finite set of functions with compact support.

We here sketch the proof by just listing a sequence of intermediate results given in [8, 9]. A preliminary characterization of $W_{-1}^{(s)}$ is given by the following proposition. Let us recall that $H^s(\mathbb{R})$ denotes the Sobolev space of functions $f \in L^2(\mathbb{R})$ such that $\dfrac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f$, with $\alpha \le s$, is element of $L^2(\mathbb{R})$.

Proposition 3. *For $s \ge 2$, $W_{-1}^{(s)}$ is the space of s^{th} derivatives of elements in $H^{2s}(\mathbb{R})$ which vanish on $\theta\Lambda$:*

$$W_{-1}^{(s)}(\Lambda) \equiv \frac{\mathrm{d}^s}{\mathrm{d}x^s} K_{2s} = \left\{ \frac{\mathrm{d}^s}{\mathrm{d}x^s} h,\ h \in H^{2s}(\mathbb{R}) \;\middle|\; \forall \lambda \in \theta\Lambda,\ h(\lambda) = 0 \right\}.$$

Proof. As indicated in Equation (12) let us determine the orthogonal complement of $V_{-1}^{(s)}(\Lambda)$ in $V_0^{(s)}(\Lambda)$. Let $f \in V_{-1}^{(s)}(\Lambda)$ and $h = \dfrac{\mathrm{d}^s}{\mathrm{d}x^s} h_1$, where $h_1 \in K_{2s}$. Then

$$\int_{\mathbb{R}} \bar{f} h \,\mathrm{d}x = \left\langle f, \frac{\mathrm{d}^s}{\mathrm{d}x^s} h_1 \right\rangle = (-1)^s \left\langle \frac{\mathrm{d}^s}{\mathrm{d}x^s} f, h_1 \right\rangle = 0.$$

Conversely, let $f \in L^2(\mathbb{R})$ and orthogonal to $W_{-1}^{(s)}(\Lambda) = \dfrac{\mathrm{d}^s}{\mathrm{d}x^s} K_{2s}$. Let us choose a function $\varphi \in C_0^\infty$ such that $\varphi(0) = 1$ and with such small compact support that its translates $\varphi(x - \lambda), \lambda \in \theta\Lambda$, have disjoint supports. For all $g \in C_0^\infty,\ g(x) - \sum_{\lambda \in \theta\Lambda} g(\lambda)\varphi(x - \lambda)$ is in K_{2s}. Accordingly

$$\int_{\mathbb{R}} \bar{f}(x) \frac{\mathrm{d}^s}{\mathrm{d}x^s} \left(g(x) - \sum_{\lambda \in \theta\Lambda} g(\lambda)\varphi(x - \lambda) \right) \mathrm{d}x = 0,$$

so we have

$$\left\langle \frac{\mathrm{d}^s}{\mathrm{d}x^s} f, g \right\rangle = \sum_{\lambda \in \theta\Lambda} \left\langle f, \frac{\mathrm{d}^s}{\mathrm{d}x^s} \varphi(x - \lambda) \right\rangle g(\lambda),$$

which means that $f \in V_{-1}^{(s)}(\Lambda)$. $\square$

Remark 1. *An equivalent characterization is to write that, for $s \ge 2$, $W_{-1}^{(s)}(\Lambda)$ is the set of functions $f \in L^2(\mathbb{R})$, $f = \dfrac{\mathrm{d}^s}{\mathrm{d}x^s} h$, for which $h \in V_0^{(2s)}(\Lambda)$ and $h|_{\theta\Lambda} = 0$.*

In view of this remark, the explicit construction of these wavelets necessitates the introduction of the following space

$$\widetilde{W}_{-1}^{(s)}(\Lambda) = \left\{ f \in V_0^{(s)}(\Lambda) \,\middle|\, f|_{\theta\Lambda} = 0 \right\}.$$

This is a closed subspace of $V_0^{(s)}(\Lambda)$. If we consider in particular $\widetilde{W}_{-1}^{(2s)}(\Lambda)$, then, by derivation $\dfrac{\mathrm{d}^s}{\mathrm{d}x^s}$ of its elements, we will obtain functions in $W_{-1}^{(s)}(\Lambda)$. Now, we need functions in $\widetilde{W}_{-1}^{(s)}(\Lambda)$ with minimal support. Let us consider the following subset of integers:

$$E = \{ n \in \mathbb{Z} \;|\; \lambda_{n+1} \notin \theta\Lambda \}. \tag{13}$$

Next, for all $n \in E$, let us define the unique function $\Psi_n^{(s)} \in \widetilde{W}_{-1}^{(s)}(\Lambda)$ satisfying the following conditions:

(i) $\operatorname{supp} \Psi_n^{(s)}$ is compact and included in $[\lambda_n, \infty)$,

(ii) $\Psi_n^{(s)}(\lambda_{n+1}) = 1$,

(iii) $f \in \widetilde{W}_{-1}^{(s)}(\Lambda),\ \operatorname{supp} f \subset \operatorname{supp} \Psi_n^{(s)} \Rightarrow f \propto \Psi_n^{(s)}$.

Then we have another important result.

Theorem 3. *The set* $\left\{ \Psi_n^{(s)},\ n \in E \right\}$ *is a Riesz basis of the space* $\widetilde{W}_{-1}^{(s)}(\Lambda)$.

The proof of this theorem is also given in [8, 9] and goes through a list of intermediate properties which we give here without proof:

Property 1. *For all $n \in E$, the support of $\Psi_n^{(s)}$ is the interval $[\lambda_n, \lambda_{n+N_n}]$, where N_n is the smallest number equal or larger than s such that* $N_n = s + \#\{(\lambda_n, \lambda_{n+N_n}) \cap \theta\Lambda\}$.

Note that N_n is also defined as the smallest number such that the interval $(\lambda_n, \lambda_{n+N_n})$ includes exactly $s-1$ points from $\Lambda \setminus \theta\Lambda$.

Property 2. *There exists a finite set $\mathcal{G}_s$ of functions with compact support such that, for all $k \in E$, $\Psi_k^{(s)}(x + \lambda_k) \in \mathcal{G}_s$.*

Property 3. *All functions from $\widetilde{W}_{-1}^{(s)}(\Lambda)$ with compact support are linear combination of the $\Psi_k^{(s)},\ k \in E$.*

Property 4. *The set of functions in* $\widetilde{W}^{(s)}_{-1}(\Lambda)$ *that have compact support is dense in* $\widetilde{W}^{(s)}_{-1}(\Lambda)$*.*

Property 5. *For all* $m \in \mathbb{Z}$*, define* I_m *as the subset of* $k \in E$ *for which* $\Psi^{(s)}_k$ *is not identically equal to zero on the interval* $[\lambda_m, \lambda_{m+1}]$*. Then* $s - 1 \leq \#(I_m) \leq s$*.*

Property 6. *There are two constants* $0 < A \leq B$*, depending on* s *and* Λ *only, such that, for any sequence* $(a_k)_{k\in E}$ *with finite support, we have*

$$A \sum_{k\in E} |a_k|^2 \leq \Big\| \sum_{k\in E} a_k \Psi^{(s)}_k \Big\|_2^2 \leq B \sum_{k\in E} |a_k|^2.$$

Note that if $A = 1 = B$ then the set $\left\{ \Psi^{(s)}_n,\ n \in E \right\}$ is an orthonormal basis. The main result of this section is a consequence of all the above statements:

Theorem 4. *(Bernuau [8, 9]) Let* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ *be a Delaunay set of finite local complexity, self-similar with factor* $\theta > 1$*. Let us denote the elements of* $\theta^{-1}\Lambda$ *by* $\kappa_n, n \in \mathbb{Z}$*,* $\kappa_n = \theta^{-1}\lambda_n$*. Then for all* $s > 1$ *there exists a Riesz basis of* $L^2(\mathbb{R})$ *of the form:*

$$\left\{ \theta^{j/2} \psi^{(s)}_{\kappa_n}(\theta^j x - \kappa_n),\ \kappa_n \in \theta^{-1}\Lambda,\ \kappa_{n+1} \notin \Lambda,\ j \in \mathbb{Z} \right\}, \tag{14}$$

where $\left\{ \psi^{(s)}_{\kappa_n}, \right\}$ *is a finite set of compactly supported functions of order* C^{s-2}*.*

Proof. For $\lambda_n \in \Lambda$, define $\zeta^{(s)}_{\lambda_n}(x - \lambda_n) = \dfrac{\mathrm{d}^s}{\mathrm{d}x^s} \Psi^{(2s)}_n(x)$. Then, according to Property 2, $\left\{ \zeta^{(s)}_{\lambda_n} \right\}$ is a finite set for $\lambda_{n+1} \notin \theta\Lambda$. These functions together with all their admissible translates $\zeta^{(s)}_{\lambda_n}(x - \lambda_n)$ form a Riesz basis of $W^{(s)}_{-1}(\Lambda)$. We know that the sequence of spaces $\left(V^{(s)}_j(\Lambda) \right)_{j\in\mathbb{Z}}$ is a multiresolution analysis of $L^2(\mathbb{R})$, so we get the orthogonal sum

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}}{}_{\perp} W^{(s)}_j(\Lambda)$$

where $f(x) \in W^{(s)}_j(\Lambda) \Longleftrightarrow f(\theta^{-j}x) \in W^{(s)}_0(\Lambda)$.

For convenience we shift by one the scale and choose the L^2-normalization in order to define

$$\psi_{\kappa_n}(x) = \frac{\zeta_{\lambda_n}(\theta x)}{\|\zeta_{\lambda_n}(\theta x)\|} \in W_0^{(s)}(\Lambda). \tag{15}$$

Then we can assert that the set $\left\{\theta^{j/2}\psi^{(s)}_{\kappa_n}(\theta^j x - \kappa_n),\ \kappa_n \in \theta^{-1}\Lambda,\ \kappa_{n+1} \notin \Lambda\right\}$ is a Riesz basis of $W_j^{(s)}(\Lambda)$. By union on all $j \in \mathbb{Z}$ we obtain the theorem. $\square$

4 Self-similar discretizations of $\mathbb{R}$ with UPV$_2$ scaling factor

4.1 Two-letter substitution sequences, quadratic PV numbers, and beta-integers

Let a be a positive integer and consider the following two types of two-letter substitution (L for "long" and S for "short")

$$\varsigma : \begin{cases} \mathrm{L} & \rightarrow \overbrace{\mathrm{L}\cdots\mathrm{L}}^{a \text{ times}}\mathrm{S}, \\ \mathrm{S} & \rightarrow \mathrm{L}, \end{cases} \quad \text{with } a \geq 1. \tag{16}$$

$$\varsigma : \begin{cases} \mathrm{L} & \rightarrow \overbrace{\mathrm{L}\cdots\mathrm{L}}^{a-1 \text{ times}}\mathrm{S}, \\ \mathrm{S} & \rightarrow \overbrace{\mathrm{L}\cdots\mathrm{L}}^{a-2 \text{ times}}\mathrm{S}, \end{cases} \quad \text{with } a \geq 3. \tag{17}$$

The corresponding substitution matrices read respectively $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a-1 & 1 \\ a-2 & 1 \end{smallmatrix}\right)$. The characteristic equation for the former is

$$X^2 - aX - 1 = 0 \text{ with roots } \begin{cases} \frac{a+\sqrt{a^2+4}}{2} \equiv \beta, \\ \frac{a-\sqrt{a^2+4}}{2} = -\frac{1}{\beta} \equiv \beta' = a - \beta. \end{cases} \tag{18}$$

For the second matrix we have

$$X^2 - aX + 1 = 0 \text{ with roots } \begin{cases} \frac{a+\sqrt{a^2-4}}{2} \equiv \beta, \\ \frac{a-\sqrt{a^2-4}}{2} = \frac{1}{\beta} \equiv \beta' = a - \beta. \end{cases} \tag{19}$$

In both cases, the largest root β is > 1 whilst its *Galois conjugate* $\beta' \in (-1, 1)$. For this reason, one says that β is a quadratic *Pisot-Vijayaraghavan*

and a unit for being invertible in their respective extension ring $\mathbb{Z}[\beta]$ (the lowest-degree coefficient in (18) and (19) is ± 1). Henceforth, we shall denote by $\mathcal{B}_-$ (resp. $\mathcal{B}_+$) the set of UPV$_2$'s obeying (18) (resp. (19)).

Let us now associate to the letter L a tile of length 1 and to S a tile of length $1/\beta$ for the case (16) and $1 - 1/\beta$ for the case (17). Starting from the origin of the real line with L on the right, we apply to it the substitution ς^∞. The set of nodes of the resulting tiling of the half-line is called the set of positive "beta-integers" and is denoted by $\mathbb{Z}_\beta^+$. Taking the symmetric $-\mathbb{Z}_\beta^+$ of $\mathbb{Z}_\beta^+$ with respect of the origin, we obtain the set of "beta-integers" $\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+)$. This set is clearly a Delaunay set which is of finite type complexity and self-similar with factor β: $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. By construction, it is also symmetrical with respect to the origin: $-\mathbb{Z}_\beta = \mathbb{Z}_\beta$. This name of *beta-integers* comes from the fact that if we retain in the set of real numbers all those numbers which are polynomial in β when written in "basis β", then they form a totally ordered discrete set that precisely coincides with $\mathbb{Z}_\beta$. We recall here that the writing of a real positive number x in irrational basis $\beta > 1$ means a unique β-expansion $x = \sum_{l=-\infty}^{j} x_l \beta^l \equiv x_j x_{j-1} \cdots x_1 x_0 \cdot x_{-1} \cdots x_l \cdots$ in which the expansion coefficients assume their values in the alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor \text{ integer part of } \beta\}$ in agreement with the existence of allowed words determined by the so-called *greedy algorithm* (for more precisions, see for instance [14, 15]). In the "minimal" case $a = 1$ of the first category (18), $\beta = \tau = (1 + \sqrt{5})/2 \approx 1.618\cdots$, the rules are particularly simple, since there the alphabet is $\{0, 1\}$ and the constraint is that no two-letter string 11 should appear in any τ-expansion words. The first pieces around the origin of the associated tiling are shown in the picture below.

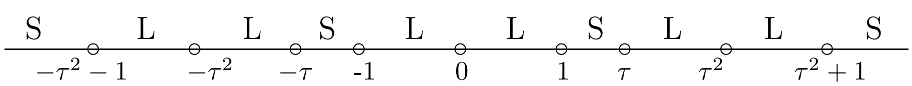

Figure 1: Tau-integers around the point 0.

Note that the case coming just next this minimal one, namely $a = 2$, is the "octogonal" number $\beta = \omega = 1 + \sqrt{2}$.

The minimal case in the second category corresponds to $a = 3$ and yields the UPV$_2$ $\beta = (3 + \sqrt{5})/2 = \tau^2 = 1 + \tau \approx 2.618\cdots$. The alphabet is now $\{0, 1, 2\}$ and the constraint is that no two-letter string $2\overbrace{1 \cdots 1}^{n\,\text{times}}2$, $n \geq 0$,

should appear in any τ^2-expansion words. The first pieces of the associated tiling on the right of the origin are shown in the picture below.

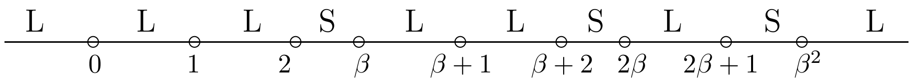

Figure 2: Beta-integers around the point 0 for $\beta = \tau^2$.

4.2 Model set discretizations of $\mathbb{R}$

Another way of obtaining self-similar Delaunay sets with finite local complexity is to resort to the so-called Cut-and-Project method which has become like a paradigm in quasicrystalline studies. We shall adopt here the formalism set up by Meyer [16, 17]. A 1+1-*cut and project scheme* is the following

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\pi_2} & \mathbb{R} \\ & & \cup & & \\ & & D & & \end{array} \tag{20}$$

where D is a lattice. The projection $\pi_1|_D$ is 1-to-1, and $\pi_2(D)$ is dense in $\mathbb{R}$.

Let $M = \pi_1(D)$ and set $^* = \pi_2 \circ (\pi_1|_D)^{-1}$

$$^* : M \longrightarrow \mathbb{R}. \tag{21}$$

The set $\Lambda \subset \mathbb{R}$ is a *model set* if there exist a cut and project scheme and a relatively compact set $\Omega \subset \mathbb{R}$ of non-empty interior such that

$$\Lambda = \{x \in M \mid x^* \in \Omega\} \equiv \Sigma^{\Omega}. \tag{22}$$

The set Ω is called a *window*.

As an illustration, we describe one type of Fibonacci chain with scaling factor $\beta = \tau^2$. Consider the cut and project scheme (20) with $D = \mathbb{Z}^2$ and $\pi_1(\mathbb{Z}^2) \sim \mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}$. The map (21) is identical up to a factor to the Galois ring automorphism $x = a + b\tau \longrightarrow x' = a - \frac{b}{\tau}$. An example of Fibonacci chain $\mathcal{F}_\tau$ [18] is given by choosing the semi-open interval $[0, 1)$ as a window.

$$\begin{aligned} \mathcal{F}_\tau &= \{x = a + b\tau \mid x' = a - \frac{b}{\tau} \in \Omega = [0,1)\} \\ &= \{\ldots, -\tau^3, -\tau, 0, \tau^2, \tau^3 + 1, \tau^4, \ldots\}. \end{aligned}$$

It is the set of left endpoints of a quasiperiodic tiling of $\mathbb{R}$ with 2 tiles L et S, of respective length τ^2 and τ, generated by the substitution rules

$$\varsigma : \begin{cases} \mathrm{L} \; \rightarrow \; \mathrm{LLS}, \\ \mathrm{S} \; \rightarrow \; \mathrm{LS}. \end{cases} \tag{23}$$

Starting from S to the left and from L to the right we get a biinfinite word

$$\cdots \mathrm{LLSLS} \mid \mathrm{LLSLLSLS} \cdots .$$

Note that $\tau^2 \mathcal{F}_\tau \subset \mathcal{F}_\tau$ and that this tiling is, by construction, *stone inflation*, which means that all the tiles when scaled by the factor τ^2 can be packed face-to-face from the original ones. Therefore, the UPV$_2$ $\tau^2 \in \mathcal{B}_+$ is the scaling factor of the substitution ς. The first pieces of this Fibonacci tiling around the origin are shown in the picture below.

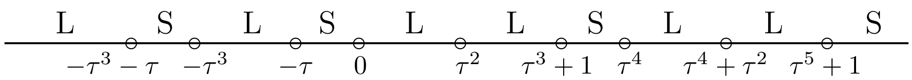

Figure 3: Fibonacci chain around the point 0.

More generally, for $\beta \in \mathcal{B}$ we shall consider the model set:

$$\mathcal{F}_\beta \equiv \{x = \mathbb{Z}[\beta] \mid x' \in \Omega = [0,1)\} \tag{24}$$

where the prime " ' " designates the ring automorphism $x = a + b\beta \longrightarrow x' = a + b\beta'$. Let us give a set of properties concerning $\mathcal{F}_\beta$ when $\beta \in \mathcal{B}$. Proofs can be found in [19, 20, 21].

Proposition 4. *Suppose that $\beta \in \mathcal{B}_-$. Then the model set $\mathcal{F}_\beta$ is a self-similar Delaunay set of finite local complexity, with scaling factor β^2. It can be characterized either as a subset of the beta-integers or as set of nodes of the stone inflation tiling associated to a substitution sequence. More precisely:*

(i) $\mathcal{F}_\beta$ is obtained from $\mathbb{Z}_\beta$ through the sieving procedure

$$\mathcal{F}_\beta = \{x \in \mathbb{Z}_\beta \mid x' \in [0,1)\}. \tag{25}$$

(ii) $\mathcal{F}_\beta$ is the set of left endpoints of a quasiperiodic tiling of $\mathbb{R}$ *with 2 tiles* L *et* S*, of respective length $\beta + 1$ and β, generated by the substitution*

rules

$$\varsigma : \begin{cases} \mathrm{L} \ \rightarrow \ \mathrm{L} \overbrace{\overbrace{\mathrm{S}\cdots\mathrm{S}}^{a-1\ times}\ \mathrm{L}\ \overbrace{\mathrm{S}\cdots\mathrm{S}}^{a-1\ times}\ \mathrm{L}\cdots\ \overbrace{\mathrm{S}\cdots\mathrm{S}}^{a-1\ times}\ \mathrm{L}}^{a\ times}\overbrace{\mathrm{S}\cdots\mathrm{S}}^{a\ times}, \\ \mathrm{S} \ \rightarrow \ \mathrm{L} \overbrace{\overbrace{\mathrm{S}\cdots\mathrm{S}}^{a-1\ times}\ \mathrm{L}\ \overbrace{\mathrm{S}\cdots\mathrm{S}}^{a-1\ times}\ \mathrm{L}\cdots\ \overbrace{\mathrm{S}\cdots\mathrm{S}}^{a-1\ times}\ \mathrm{L}}^{a-1\ times}\overbrace{\mathrm{S}\cdots\mathrm{S}}^{a\ times}, \end{cases} \tag{26}$$

with $a \geq 1$. The tiling is obtained by starting from S*-origin-*L *in both directions.*

Proposition 5. *Suppose that $\beta \in \mathcal{B}_+$. Then the model set $\mathcal{F}_\beta$ is a self-similar Delaunay set of finite local complexity, with scaling factor β. It can be characterized either as a subset of decorated beta-integers or as set of nodes of the stone inflation tiling associated to a substitution sequence. More precisely:*

(i) $\mathcal{F}_\beta$ is obtained from the decorated beta-integers $\widetilde{\mathbb{Z}}_\beta \stackrel{def}{=} \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\}$ through the sieving procedure

$$\mathcal{F}_\beta = \{x \in \widetilde{\mathbb{Z}}_\beta \mid x' \in [0,1)\}. \tag{27}$$

(ii) Alternatively, $\mathcal{F}_\beta$ reads as

$$\mathcal{F}_\beta = \beta\left[\mathbb{Z}_\beta^+ \cup \left((-\mathbb{Z}_\beta^+) \setminus \{0\} + \frac{1}{\beta}\right)\right]. \tag{28}$$

(iii) $\mathcal{F}_\beta$ is the set of left endpoints of the quasiperiodic tiling of $\mathbb{R}$ with 2 tiles L *et* S*, of respective length β and $\beta - 1$, generated by the substitution rules* (17)*. The tiling is obtained by starting from* S*-origin-*L *in both directions.*

Note that the minimal case $\beta = \tau$ is exceptional in the sense that there is equality between both extension rings $\mathbb{Z}[\tau] = \mathbb{Z}[\tau^2]$ and so one can deduces from (28): $\mathcal{F}_\tau = \beta\left[\mathbb{Z}_\beta^+ \cup \left((-\mathbb{Z}_\beta^+) \setminus \{0\} + \frac{1}{\beta}\right)\right]$, with $\beta = \tau^2$.

5 Haar wavelets for β-integers

We now turn our attention on the construction of multiresolution analysis of the Haar type based on non-negative β-integers $\mathbb{Z}_\beta^+$, the extension to the full $\mathbb{Z}_\beta$ being carried out by simple symmetry with respect to the origin. Let us denote by space V_0^+ the closure in $L^2(\mathbb{R}^+)$ of the linear span of all positive admissible translates of normalized characteristic functions $\phi_{\mathrm{L}}(x)$ and $\phi_{\mathrm{S}}(x)$ supported by intervals of lengths $|\mathrm{L}|$ and $|\mathrm{S}|$ respectively. More precisely,

$$V_0^+ = \overline{\mathrm{span}\left\{\phi_{\mathrm{L}}(x-\lambda_{\mathrm{L}}), \phi_{\mathrm{S}}(x-\lambda_{\mathrm{S}})\right\}_{\lambda_{\mathrm{L}}\in\Lambda_{\mathrm{L}}^+, \lambda_{\mathrm{S}}\in\Lambda_{\mathrm{S}}^+}},$$

where Λ_{L}^+ is the set of left-hand ends of tiles L in $\mathbb{Z}_\beta^+$ and Λ_{S}^+ is the set of left-hand ends of tile S in $\mathbb{Z}_\beta^+$.

Two possibilities exist for constructing basis made up with piecewise constant functions. Either one builds orthonormal basis, at the price of increasing the number of *mother* functions, or one just requires a Riesz basis and then we need two *mother* functions only.

5.1 Orthogonal basis of Haar type

Let us first discuss the case (16). Due to the two-letter substitution rules, two possible distances between points exist in the associated tiling, namely $|\mathrm{L}| = 1$ and $|\mathrm{S}| = 1/\beta$. Consistently, two scaling functions exist, one per type of tile, and their refinement equations are precisely based on these substitution rules. For $a \geq 1$ the two scaling functions read:

$$\phi_{\mathrm{L}}(x) = \mathbf{1}_{[0,1)}(x), \tag{29}$$

$$\phi_{\mathrm{S}}(x-a) = \beta^{1/2}\mathbf{1}_{[a,a+1/\beta)}(x). \tag{30}$$

Consequently, the scaling (or refinement) equations for $a \geq 2$ are the following:

$$\phi_{\mathrm{L}}(x) = \sum_{l=0}^{a-1}\phi_{\mathrm{L}}(\beta x - l) + \beta^{-1/2}\phi_{\mathrm{S}}(\beta x - a), \tag{31}$$

$$\phi_{\mathrm{S}}(x-a) = \beta^{1/2}\phi_{\mathrm{L}}(\beta x - a\beta). \tag{32}$$

The corresponding orthonormal wavelet basis is built from a mother wavelets $\{\psi_{\mathrm{L},i}\}_{i=0}^{a-1}$. One method of construction of these functions is to

Gram-Schmidt orthogonalize and normalize in the sense of L^2-norm the set of functions $\{\phi_{\mathrm{L}}(x), \phi_{\mathrm{L}}(\beta x), \phi_{\mathrm{L}}(\beta x-1), \ldots, \phi_{\mathrm{L}}(\beta x-(a-1))\}$ living on the interval $[0,1]$. An orthonormal basis for $V_1^+ = V_0^+ \oplus_\perp W_0^+$ is then obtained by collecting together all translates of our orthonormal set at points of Λ_{L}^+ and all translates of the ϕ_{S}'s at points of Λ_{S}^+. After removing functions in V_0^+ we get the basis of W_0^+.

The case $a=1$, $\beta=\tau$ is particularly interesting. In this case, $\Lambda_{\mathrm{L}}^+ = \tau\mathbb{Z}_\tau^+$ and $\Lambda_{\mathrm{S}}^+ = \tau^2\mathbb{Z}_\tau^+ + 1$. There are two scaling functions but only one wavelet, and so the refinement equations read as:

$$\begin{aligned} \phi_{\mathrm{L}}(x) &= \phi_{\mathrm{L}}(\tau x) + \tau^{-1/2}\phi_{\mathrm{S}}(\tau x-1), \\ \phi_{\mathrm{S}}(x-1) &= \tau^{1/2}\phi_{\mathrm{L}}(\tau x-\tau), \\ \psi_{\mathrm{L}}(x) &= \tau^{-1/2}\phi_{\mathrm{L}}(\tau x) - \phi_{\mathrm{S}}(\tau x-1). \end{aligned} \tag{33}$$

Note that in the present case the following conditions are equivalent:

$$\lambda_n \in \tau^{-1}\mathbb{Z}_\tau^+ \text{ and } \lambda_{n+1} \notin \mathbb{Z}_\tau^+ \Leftrightarrow \lambda_n \in \tau^{-1}\Lambda_{\mathrm{LS}}^+ \Leftrightarrow \lambda_n \in \tau\mathbb{Z}_\tau^+ \equiv \Lambda_{\mathrm{L}}^+, \tag{34}$$

where we have denoted by λ the generic elements of $\tau^{-1}\mathbb{Z}_\tau^+$. Consequently, the orthonormal basis of V_1^+ is given by:

$$\{\phi_{\mathrm{S}}(x-\lambda)\}_{\lambda\in(\tau^2\mathbb{Z}_\tau^+ +1)} \cup \{\phi_{\mathrm{L}}(x-\lambda)\}_{\lambda\in\tau\mathbb{Z}_\tau^+} \cup \{\psi_{\mathrm{L}}(x-\lambda)\}_{\lambda\in\tau\mathbb{Z}_\tau^+}, \tag{35}$$

and the orthonormal ("tau") Haar basis of $L^2(\mathbb{R}^+)$ is the set

$$\left\{\tau^{j/2}\psi_{\mathrm{L}}(\tau^j x-\lambda)\right\}_{j\in\mathbb{Z},\lambda\in\tau\mathbb{Z}_\tau^+}. \tag{36}$$

The construction of the Haar wavelet basis corresponding to the substitution of the second type is carried out in a similar way. Lengths of tiles are respectively $|\mathrm{L}|=1$ and $|\mathrm{S}| = 1-1/\beta$. For $a\geq 3$ we have for the two scaling functions:

$$\phi_{\mathrm{L}}(x) = \mathbf{1}_{[0,1)}(x), \tag{37}$$

$$\phi_{\mathrm{S}}(x-a+1) = (1-1/\beta)^{-1/2}\mathbf{1}_{[a-1,a-1/\beta)}(x). \tag{38}$$

Consequently, the refinement equations now read:

$$\phi_{\mathrm{L}}(x) = \sum_{l=0}^{a-2}\phi_{\mathrm{L}}(\beta x-l) + (1-1/\beta)^{1/2}\phi_{\mathrm{S}}(\beta x-(a-1)), \tag{39}$$

$$\begin{aligned} \phi_{\mathrm{S}}(x-a+1) = (1-1/\beta)^{-1/2}\sum_{l=0}^{a-3}&\phi_{\mathrm{L}}(\beta x-\beta(a-1)-l)+ \\ &+\phi_{\mathrm{S}}(\beta x-(a-1)\beta-a+2). \end{aligned} \tag{40}$$

The corresponding orthonormal wavelet basis is constructed from the mother wavelet sets $\{\psi_{\mathrm{L},i}(x)\}_{i=0}^{a-2}$ and $\{\psi_{\mathrm{S},i}(x)\}_{i=0}^{a-3}$. Following the same method as in the previous case, we first pick the functions $\{\phi_{\mathrm{L}}(x), \phi_{\mathrm{L}}(\beta x), \phi_{\mathrm{L}}(\beta x - 1), \ldots, \phi_{\mathrm{L}}(\beta x-(a-1))\}$ living on the interval $[0,1]$ and $\{\phi_{\mathrm{S}}(x), \phi_{\mathrm{L}}(\beta x), \phi_{\mathrm{L}}(\beta x - 1), \ldots, \phi_{\mathrm{L}}(\beta x-(a-2))\}$ living on $[0, 1-1/\beta]$. We then proceed to the Gram-Schmidt orthogonalization and L^2-normalization. The orthonormal basis of V_1^+ is obtained by collecting together all translates of the first orthonormal set at points of Λ_{L}^+ and all translates of the second orthonormal set at points of Λ_{S}^+. After removing functions in V_0^+ we get the basis of W_0^+.

For the negative part of $\mathbb{Z}_\beta$ we just have to mirror the supports of scaling functions and wavelets.

5.2 Non-orthogonal basis of Haar type

We still have same scaling functions as in the orthogonal case. This means that all spaces V_j^+ are identical to the previous ones, but the basis of the orthogonal complement W_0^+ of V_0^+ in V_1^+ is just required to be normalized. But the case $\beta = \tau$ which leads to the same result as in the above, there are generically two mother wavelets ψ_{LL} and ψ_{LS} which live on tiles β^{-1}LL, resp. β^{-1}LS. The orthogonal complement of V_0^+ in V_1^+ reads:

$$W_0^+ = \overline{\mathrm{span}\,\{\psi_{\mathrm{LL}}(x-\lambda_{\mathrm{LL}}), \psi_{\mathrm{LS}}(x-\lambda_{\mathrm{LS}})\}_{\lambda_{\mathrm{LL}}\in\beta^{-1}\Lambda_{\mathrm{LL}}^+, \lambda_{\mathrm{LS}}\in\beta^{-1}\Lambda_{\mathrm{LS}}^+}},$$

where Λ_{LL}^+ (resp. Λ_{LS}^+) is the set of left-hand ends of words LL (resp. LS) in $\mathbb{Z}_\beta^+$.

In the substitutional case (16) the refinement equations for wavelets are:

$$\begin{aligned}
\psi_{\mathrm{LL}}(x) &= \left(\frac{\beta}{2}\right)^{1/2} (\phi_{\mathrm{L}}(\beta x) - \phi_{\mathrm{L}}(\beta x - 1)), \\
\psi_{\mathrm{LS}}(x-(a-1)/\beta) &= \left(\frac{\beta}{\beta+1}\right)^{1/2} (\phi_{\mathrm{L}}(\beta x - (a-1)) - \beta^{1/2}\phi_{\mathrm{S}}(\beta x - a)).
\end{aligned}$$

The construction of the Haar wavelet basis corresponding to the substitution of the second type is carried out in a similar way. The corresponding

wavelets are given by:

$$\psi_{\mathrm{LL}}(x) = \left(\frac{\beta}{2}\right)^{1/2} (\phi_{\mathrm{L}}(\beta x) - \phi_{\mathrm{L}}(\beta x - 1)),$$

$$\psi_{\mathrm{LS}}(x - (a-2)/\beta) =$$

$$\left(\frac{(a-1)\beta - 1}{2\beta - 1}\right)^{1/2} (\phi_{\mathrm{L}}(\beta x - a + 2) - (1 - 1/\beta)^{-1/2}\phi_{\mathrm{S}}(\beta x - a + 1)).$$

For the negative part of $\mathbb{Z}_\beta$ scaling functions and their equations are still the same but instead of the wavelet ψ_{LS} we now have to deal with a wavelet of the type ψ_{SL}.

6 Lexicographical analysis of the Fibonacci chain

We now focus on the rescaled version $\Lambda \stackrel{\mathrm{def}}{=} \mathcal{F}_\tau/\tau^2$ of the Fibonacci chain $\mathcal{F}_\tau$. This is a standard example of model set, well known by quasicrystallographers, and most of the obtained results about it are easily extendable to other cases described in Section 4. With the notation (22) this model set reads $\Lambda = \Sigma^{[0,\tau^2)}$. We also view $\Lambda \equiv \{\lambda_n\}_{n\in\mathbb{Z}}$ as the increasing sequence of points $\lambda_n < \lambda_{n+1}$ with $\lambda_0 = 0$. Let us give a list of important properties stemming from the interval nature of the window:

- The model set Λ is self-similar with factor τ^2: $\tau^2\Lambda \subset \Lambda$. More precisely, it is invariant under the affine-linear actions:

$$\tau^2\Lambda = \Sigma^{[0,1)} \subset \Lambda, \quad \tau^2\Lambda + \Sigma^{[0,\tau]} = \Lambda. \tag{41}$$

- For a given n, Λ reads as a partition into subsets of n-letter words $\Lambda_{a_1a_2\ldots a_n} = \{\lambda\,|\,\lambda$ is the left-hand end of the n-letter word $a_1a_2\ldots a_n$, $a_i \in \{\mathrm{L},\mathrm{S}\}\}$. More precisely, we have the following tree-like *Fibonacci* hierarchy of partitions:

$$\begin{aligned}
\Lambda &= \Lambda_{\mathrm{L}} \cup \Lambda_{\mathrm{S}}, \\
\Lambda &= \overbrace{\Lambda_{\mathrm{LL}} \cup \Lambda_{\mathrm{LS}}} \cup \Lambda_{\mathrm{SL}} \\
\Lambda &= \Lambda_{\mathrm{LLS}} \cup \Lambda_{\mathrm{LSL}} \cup \overbrace{\Lambda_{\mathrm{SLL}} \cup \Lambda_{\mathrm{SLS}}} \\
&\vdots
\end{aligned}$$

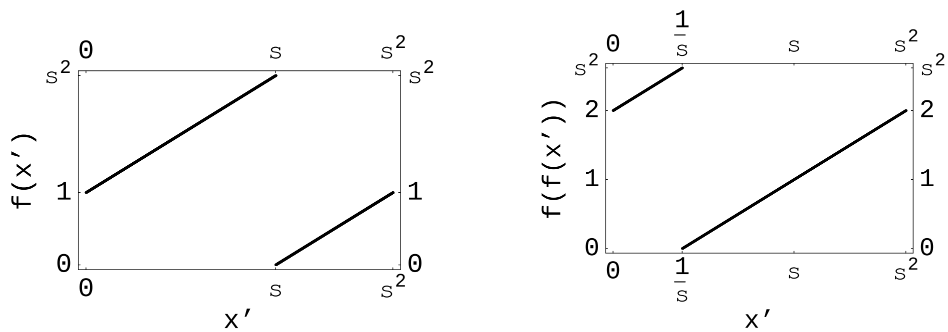

Figure 4: Conjugate right nearest neighbour function f (left graph) and its iterated $f \circ f$ (right graph).

There are only $n+1$ words built up from n letters (the substitution is Sturmian with *minimal complexity*).

- The above sequence of partitions is in one-to-one correspondence with partitions of the interval window $[0, \tau^2)$. Accordingly, each element in the partition sequence is also a model set: $\Lambda_{a_1 a_2 \dots a_n} = \Sigma^{[\omega_1, \omega_2)}$, where $\omega_1, \omega_2 \in \mathbb{Z}[\tau]$.

The properties of sets $\Lambda_{a_1 a_2 \dots a_n}$, are suitably encoded by the one-to-one map $f : \Lambda' \longrightarrow \Lambda'$ introduced in [21] and defined by $f(\lambda'_n) = \lambda'_{n+1}$ where $\lambda_n \in \Lambda$. This function uniquely determines the nearest right neighbour of any point $\lambda \in \Lambda$. Its reciprocal $f^{-1}(\lambda'_n) = \lambda'_{n-1}$ corresponds to the nearest left neighbour. The function f is explicitly given by:

$$f(x') = \begin{cases} x' + 1 & \text{if } x \in \Lambda_{\mathrm{L}} \text{ (since } \lambda_{n+1} = \lambda_n + 1), \\ x' - \tau & \text{if } x \in \Lambda_{\mathrm{S}} \text{ (since } \lambda_{n+1} = \lambda_n + 1/\tau). \end{cases} \tag{42}$$

The graphs of the function $f(x')$ and $f(f(x'))$ are shown in Fig. 4.

The function f divides the interval $[0, \tau^2)$ into two parts. The first one is the interval $[0, \tau)$, window of Λ_{L}, and the second is $[\tau, \tau^2)$, window of Λ_{S}. As one can see in Fig. 4 the next right neighbour of the conjugate $\tau' = -1/\tau$ of the discontinuity point τ is the origin $0' = 0$. Next, let us plot $f^{[2]}(x) \equiv f(f(x))$. This function encodes the distance to the second next right neighbour. The interval $[0, \tau^2)$ is again split into two parts. This also means that 2-letter words have only two possible lengths. Generally, for each $f^{[n]}$ we will find a new point of discontinuity which splits the interval $[0, \tau^2)$

into two parts and the n^{th} next right neighbour of its Galois conjugate is the point 0. All points lying on the segment common to all iterates of f up to the n^{th} one are the conjugates of left-hand ends of the same n-letter word – they induce a model set for this n-letter word. The following proposition easily ensues from this observation:

Proposition 6. *Left-hand ends of all possible n-letter words lie as the closest n left neighbours of the origin together with the latter, namely the points $\lambda_{-n}, \ldots, \lambda_0$.*

Proof. We know from the above that all these points are in different model sets and since there are $n+1$ possible words we know that no one is missing. ☐

If we rearrange from the largest to the smallest one all n-letter words in the lexicographical order determined by L $>$ S, $\Lambda_0^{(n)} > \Lambda_1^{(n)} > \cdots > \Lambda_{n-1}^{(n)} > \Lambda_n^{(n)}$, we find that $\Lambda_k^{(n)} = \Sigma^{[{\lambda_k^{(n)}}', {\lambda_{k+1}^{(n)}}')}$ for $k = 0, 1, \ldots, n$, with $\Lambda_n^{(n)} = \Sigma^{[{\lambda_n^{(n)}}', \tau^2)}$. The $\lambda_k^{(n)}$'s $\in \{\lambda_{-n}, \ldots, \lambda_0\}$ are left-hand ends of $\Lambda_k^{(n)}$ and are ordered as ${\lambda_k^{(n)}}' < {\lambda_{k+1}^{(n)}}'$.

Let us consider a n-tuple $(\lambda_k, \lambda_{k+1}, \ldots, \lambda_{k+n}) \in \Lambda^n$. We know that $\lambda_{k+i} - \lambda_{k+i-1} = 1$ or $1/\tau$. Consequently we have

$$\tau^2(\lambda_{k+i} - \lambda_{k+i-1} - 1/\tau) = \begin{cases} 0 & \text{if } \lambda_{k+i} \text{ and } \lambda_{k+i-1} \text{ are ends of tile S,} \\ 1 & \text{if } \lambda_{k+i} \text{ and } \lambda_{k+i-1} \text{ are ends of tile L,} \end{cases}$$

which implies the following property.

Property 7. *The number of* L*'s in the word $[\lambda_k, \lambda_{k+n}]$ is equal to*

$$\sum_{i=1}^{n} \tau^2(\lambda_{k+i} - \lambda_{k+i-1} - 1/\tau) = \tau^2(\lambda_{k+n} - \lambda_k - n/\tau).$$

Let us now characterize all points of $\Sigma^{[0,1)} = \tau^2 \Sigma^{[0,\tau^2)}$ in another way. From

$$\Sigma^{[0,1)} = \left\{ l + k\tau \mid l, k \in \mathbb{Z}, l - \frac{k}{\tau} \in [0, 1) \right\} \Longrightarrow \frac{k}{\tau} \leq l \leq \frac{k}{\tau} + 1 \Longrightarrow l = \left\lceil \frac{k}{\tau} \right\rceil,$$

we easily recover a standard definition for chains of the Fibonacci type [22]:

Property 8.

$$\Sigma^{[0,\tau^2)} \equiv \frac{1}{\tau^2}\Sigma^{[0,1)} \equiv \left\{ \lambda_k \;\middle|\; \lambda_k = \frac{1}{\tau^2}\left(\left\lceil \frac{k}{\tau} \right\rceil + k\tau\right),\ k \in \mathbb{Z} \right\}.$$

It follows that the number of L's in any n-letter word is

$$\left(\left\lceil \frac{k+n}{\tau} \right\rceil + (k+n)\tau - \left\lceil \frac{k}{\tau} \right\rceil - k\tau - n\tau\right) = \left\lceil \frac{k+n}{\tau} \right\rceil - \left\lceil \frac{k}{\tau} \right\rceil =$$

$$= \left\lceil \frac{k+n}{\tau} - \left\lceil \frac{k}{\tau} \right\rceil \right\rceil = \left\lceil \frac{n}{\tau} + \frac{k}{\tau} - \left\lceil \frac{k}{\tau} \right\rceil \right\rceil = \left\lceil \frac{n}{\tau} \right\rceil \text{ or } \left\lfloor \frac{n}{\tau} \right\rfloor.$$

Accordingly the number of S's in any n-letter word is $n - \left\lceil \frac{n}{\tau} \right\rceil = \left\lfloor \frac{n}{\tau^2} \right\rfloor$ or $n - \left\lfloor \frac{n}{\tau} \right\rfloor = \left\lceil \frac{n}{\tau^2} \right\rceil$. In summary:

Property 9. *In any n-letter word there are $\left\lceil \frac{n}{\tau} \right\rceil$ L's and $\left\lfloor \frac{n}{\tau^2} \right\rfloor$ S's or $\left\lfloor \frac{n}{\tau} \right\rfloor$ L's and $\left\lceil \frac{n}{\tau^2} \right\rceil$ S's.*

7 Fibonacci B-splines and wavelets

7.1 B-splines scaling functions for the Fibonacci chain

Using Def. 1 we have for $s \geq 2$,

$$V_0^{(s)}(\Lambda) = \left\{ f \in C^{s-2}, f \in L^2(\mathbb{R}) \;\middle|\; f|_{[\lambda_n,\lambda_{n+1}]} \text{ is polynomial of degree} \leq s-1 \right\},$$

where we remind that λ_n designates the n^{th} element of Λ with $\lambda_0 = 0$. The Riesz basis of $V_0^{(s)}(\Lambda)$ is made up of translates of $s+1$ functions ϕ_{Ω_i}, where the Ω_i's, $i = -s, \ldots, 0$, are all admissible s-letter words. Let us suppose that the support corresponding to the word Ω_i is the interval $[\lambda_i, \lambda_{i+s}]$. Then we construct all s functions $\phi_{\Omega_i} \equiv \phi_i$ through the conditions below.

- $\phi_i(x - \lambda_i) = 0$ if $x \notin (\lambda_i, \lambda_{i+s})$.
- $\phi_i(x - \lambda_i)$ is polynomial of degree $s-1$ on the intervals $[\lambda_{i+k}, \lambda_{i+k+1}]$, for $0 \leq k \leq s-1$.
- The $\phi_i^{(l)}(x - \lambda_i)$'s, $0 \leq l \leq s-2$, are continuous in the points λ_{i+k} for $0 \leq k \leq s$.
- $\displaystyle\int_{\mathbb{R}} \phi_i(x)\,\mathrm{d}x = \frac{\lambda_{i+s} - \lambda_i}{s}$.

7.2 Spline wavelets for the Fibonacci chain

The construction of the wavelet basis of $W_{-1}^{(s)}(\Lambda)$ rests upon the construction of the basis $\left\{\Psi_n^{(2s)},\ n \in E\right\}$ of the space $\widetilde{W}_{-1}^{(2s)}(\Lambda)$. The first question to answer is how many different wavelet functions we have to build up and what are their supports. From (13) and Fig. 4 we see that $\Lambda_E = \{\lambda_n \mid n \in E\} = \{\lambda_n \in \Lambda \mid \lambda_{n+1} \notin \tau^2\Lambda\} = \Lambda_{\mathrm{L}}$. Hence we know that all supports start with L. The answer about the lengths of supports is afforded by Property 1. We recall that the support of the function $\Psi_k^{(2s)},\ k \in E$, is the interval $(\lambda_k, \lambda_{k+N_k})$ which includes exactly $2s-1$ points from $\Lambda\backslash\tau^2\Lambda$ and is the smallest possible. Let us now observe a few simple facts. We want to find all possible minimal supports containing $2s-1$ point of $\Lambda\backslash\tau^2\Lambda$. We know that any n-letter word supports $n-1$ points of the Fibonacci chain. Since $\tau^2\Lambda = \Lambda_{\mathrm{S}} + 1/\tau$, the number of points of $\tau^2\Lambda$ in the interval (word) $(\lambda_k, \lambda_{k+n})$ is the same as the number of S's in the word $\langle\lambda_k, \lambda_{k+n-1}\rangle$. If $\left\lceil\frac{n-1}{\tau^2}\right\rceil$ or $\left\lfloor\frac{n-1}{\tau^2}\right\rfloor$ is the number of points of $\tau^2\Lambda$ in the interval $(\lambda_k, \lambda_{k+n})$ then $\left\lfloor\frac{n-1}{\tau}\right\rfloor$ or $\left\lceil\frac{n-1}{\tau}\right\rceil$ is the number of points of $\Lambda\backslash\tau^2\Lambda$ in the same interval. We now prove the following:

Proposition 7. *Let p be the length of the shortest word supporting $2s-1$ points of $\Lambda\backslash\tau^2\Lambda$. Then all other minimal words starting with* L *supporting $2s-1$ points of $\Lambda\backslash\tau^2\Lambda$ have their lengths equal to p or $p+1$.*

Proof. In words of length p there are $\left\lceil\frac{p-1}{\tau}\right\rceil = 2s-1$ or $\left\lfloor\frac{p-1}{\tau}\right\rfloor = 2s-2$ points of $\Lambda\backslash\tau^2\Lambda$. The former words are precisely those we wish to find while we have to enlarge the latter with one point more. A successor (a $(p+1)$-letter word which has the same p first letters) to words starting with L of the second type has to support $2s-1$ points from $\Lambda\backslash\tau^2\Lambda$ also. Indeed, if it had $2s-2$ points from $\Lambda\backslash\tau^2\Lambda$ only, we could remove the first L (and this removes also one point from $\Lambda\backslash\tau^2\Lambda$). Then we would have a p-letter word supporting $2s-3$ points from $\Lambda\backslash\tau^2\Lambda$ and this is not possible. □

As it has been shown in the above, all words of length $l = \max_{k\in E} N_k$ (from Proposition 7 we also know that $l-1 = \min_{k\in E} N_k$) can be found among those having left-hand end in the interval $[\lambda_{-l}, \lambda_0]$. From the definition of N_k one can infer that there are $2s-1$ wavelets which overlap each other in the interval $[\lambda_0, \lambda_1]$. These wavelets are certainly all different – they have different supports. But the first wavelet the support of which starts from λ_{-l} or from λ_{-l+1} (if $\lambda_{-l+1} \in \tau^2\Lambda$) ends at λ_0 (it cannot end at λ_1 because $\lambda_0 \in \tau^2\Lambda$ and also not at λ_2 because then the length is larger than l). Consequently, we

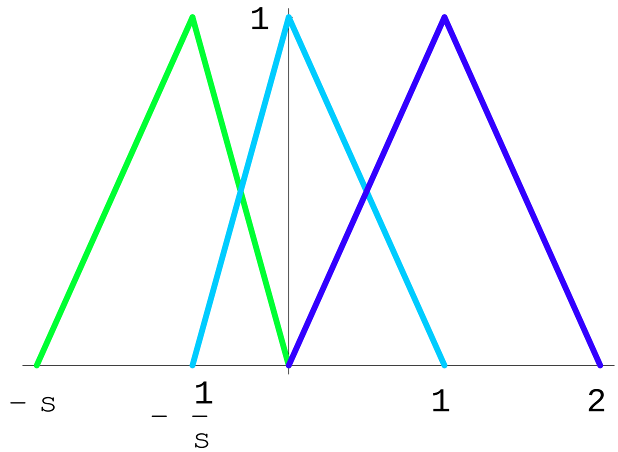

Figure 5: Linear scaling functions $\phi_{\mathrm{LS}}(x+\tau)$, $\phi_{\mathrm{SL}}(x+\frac{1}{\tau})$ and $\phi_{\mathrm{LL}}(x)$.

have to select this wavelet also. Hence the total number of different wavelets is $2s$ and we can finally assert the following:

Proposition 8. *For a given s there are exactly $s+1$ B-spline scaling functions and $2s$ different wavelets. The supports of the latter have length equal to $p = \lceil (2s-2)\tau \rceil + 1$ or to $p+1$.*

7.3 Second-order splines for the Fibonacci chain

From Fig. 3 and also from the underlying substitution rules, we can see that for $s=2$, there are just 3 possible words, namely LL, LS, SL. Hence we get the following scaling functions:

$$\phi_{\mathrm{LL}}(x) = \begin{cases} x & x \in [0,1], \\ 2-x & x \in [1,2], \end{cases}$$

$$\phi_{\mathrm{LS}}(x+\tau) = \begin{cases} x+\tau & x \in [-\tau, -1/\tau], \\ \tau^2 - \tau(x+\tau) & x \in [-1/\tau, 0], \end{cases}$$

$$\phi_{\mathrm{SL}}(x+1/\tau) = \begin{cases} \tau(x+1/\tau) & x \in [-1/\tau, 0], \\ \tau - (x+1/\tau) & x \in [0,1]. \end{cases}$$

The scaling equations are:

$$
\begin{aligned}
\phi_{\mathrm{LL}}(x) = {} & \frac{1}{\tau^2}\phi_{\mathrm{LL}}(\tau^2 x) + \frac{2}{\tau^2}\phi_{\mathrm{LS}}(\tau^2 x - 1) + \phi_{\mathrm{SL}}(\tau^2 x - 2) + \\
& \frac{1}{\tau}\phi_{\mathrm{LL}}(\tau^2 x - \tau^2) + \frac{1}{\tau^3}\phi_{\mathrm{LS}}(\tau^2 x - \tau^2 - 1), \\
\phi_{\mathrm{LS}}(x+\tau) = {} & \frac{1}{\tau^2}\phi_{\mathrm{LL}}(\tau^2 x + \tau^3) + \frac{2}{\tau^2}\phi_{\mathrm{LS}}(\tau^2 x + \tau^2 + 1/\tau) + \\
& \phi_{\mathrm{SL}}(\tau^2 x + \tau + 1/\tau) + \frac{1}{\tau^2}\phi_{\mathrm{LS}}(\tau^2 x + \tau), \\
\phi_{\mathrm{SL}}(x+1/\tau) = {} & \frac{1}{\tau}\phi_{\mathrm{LS}}(\tau^2 x + \tau) + \phi_{\mathrm{SL}}(\tau^2 x + 1/\tau) + \frac{1}{\tau}\phi_{\mathrm{LL}}(\tau^2 x) + \\
& \frac{1}{\tau^3}\phi_{\mathrm{LS}}(\tau^2 x - 1).
\end{aligned}
$$

So in this case the Riesz basis of $V_0^{(2)}(\Lambda)$ is the set of functions

$$
\{\phi_{\mathrm{LL}}(x-\lambda_{\mathrm{LL}}),\, \phi_{\mathrm{LS}}(x-\lambda_{\mathrm{LS}}),\, \phi_{\mathrm{SL}}(x-\lambda_{\mathrm{SL}})\}_{\lambda_{\mathrm{LL}}\in\Lambda_{\mathrm{LL}},\lambda_{\mathrm{LS}}\in\Lambda_{\mathrm{LS}},\lambda_{\mathrm{SL}}\in\Lambda_{\mathrm{SL}}}\,.
$$

In order to make the explicit computation of wavelets more transparent we recall in Table 1 below the points of the Fibonacci chain $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ around zero.

L	L	S	L	S	L	L	S	L	L	S
λ_{-5}	λ_{-4}	λ_{-3}	λ_{-2}	λ_{-1}	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
$-\tau^3$	$-\tau^2-\frac{1}{\tau}$	$-\tau-\frac{1}{\tau}$	$-\tau$	$-\frac{1}{\tau}$	0	1	$\tau+\frac{1}{\tau^2}$	τ^2	τ^2+1	$\tau^3+\frac{1}{\tau^2}$
$-1-2\tau$	-2τ	$1-2\tau$	$-\tau$	$1-\tau$	0	1	2	$1+\tau$	$2+\tau$	$3+\tau$
yes	no	no	yes	no	yes	no	no	yes	no	no

Table 1: Fibonacci chain points located around the origin.

The first row tells us whether the point is a left-hand end of the tile L or S, the second row is just the indexation of points, in the third row there points of the Fibonacci chain are written in terms of τ-expansion, in the fourth row they are written in the form $a+b\tau$. The last row indicates whether the point is element of $\tau^2\Lambda$ or not.

Let us now construct the wavelet basis of $W_0^{(2)}(\Lambda)$ by applying the results of Section 3. We look at first for the $\Psi_n^{(2s=4)}$'s, $n \in E$, which form the basis of $\widetilde{W}_{-1}^{(4)}(\Lambda) = \left\{ f \in V_0^{(4)} \mid f|_{\tau^2\Lambda} = 0 \right\}$. Their respective supports are $[\lambda_n, \lambda_{n+N_n}]$, where $n \in E \equiv \{n \in \mathbb{Z} \mid \lambda_{n+1} \notin \tau^2\Lambda\}$ and N_n is the smallest number for which the equality $N_n = 4 + \#\{(\lambda_n, \lambda_{n+N_n}) \cap \tau^2\Lambda\}$

tile	interval	$\zeta_{\mathrm{LLSLS}}(x-\lambda_{-5})$		$\zeta_{\mathrm{LSLSLL}}(x-\lambda_{-4})$	
		k_i	q_i	k_i	q_i
L	$[\lambda_{-5},\lambda_{-4})$	6	0		
L	$[\lambda_{-4},\lambda_{-3})$	$-\frac{6(1+26\tau)}{11}$	6	6	0
S	$[\lambda_{-3},\lambda_{-2})$	$\frac{12(18+17\tau)}{11}$	$\frac{12(5-13\tau)}{11}$	$-24(1+2\tau)$	6
L	$[\lambda_{-2},\lambda_{-1})$	$-\frac{24(1+4\tau)}{11}$	$\frac{12(4+5\tau)}{11}$	$\frac{6(91+123\tau)}{11}$	$-6(3+4\tau)$
S	$[\lambda_{-1},\lambda_{0})$	$\frac{12(3+\tau)}{11}$	$\frac{12(2-3\tau)}{11}$	$-\frac{6(127+224\tau)}{11}$	$\frac{6(58+79\tau)}{11}$
L	$[\lambda_{0},\lambda_{1})$			$\frac{21(13+16\tau)}{11}$	$-\frac{18(13+16\tau)}{11}$
L	$[\lambda_{1},\lambda_{2})$			$-\frac{3(13+16\tau)}{11}$	$\frac{3(13+16\tau)}{11}$
norm		$\frac{2}{11}\sqrt{6(270+431\tau)}\doteq 13.8519$		$\frac{2}{11}\sqrt{3(16754+27145\tau)}\doteq 77.572$	

Table 2: Analytical expression of $\zeta_{\mathrm{LLSLS}}(x-\lambda_{-5})$ and $\zeta_{\mathrm{LSLSLL}}(x-\lambda_{-4})$.

holds true. We recall that the equality $\Lambda_E = \{\lambda_n \in \Lambda \mid n \in E\} = \Lambda_{\mathrm{L}}$ determines that all supports will start by L. The set of these supports is $\Omega_\psi = \{\mathrm{LLSLS, LSLSLL, LSLLS, LLSLL}\}$. This means that we have four different shapes of $\Psi_n^{(4)}$'s, one per support. As second derivatives of functions $\Psi_n^{(4)}(x)$, $n \in E$ we obtain four different functions $\zeta_{\lambda_n}(x)$, precisely $\zeta_{\lambda_n}(x-\lambda_n) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_n^{(4)}(x)$, $n \in E$ (see Fig. 6). Due to the equality $\zeta_{\lambda_n}(x) = \zeta_{\lambda_k}(x)$ if $\lambda_n,\ \lambda_k \in \Lambda_\mu$ where $\mu \in \Omega_\psi$, we just denote the latter by $\zeta_\mu(x)$.

From Table 2 and 3 we deduce the analytical expressions of functions ζ_μ,

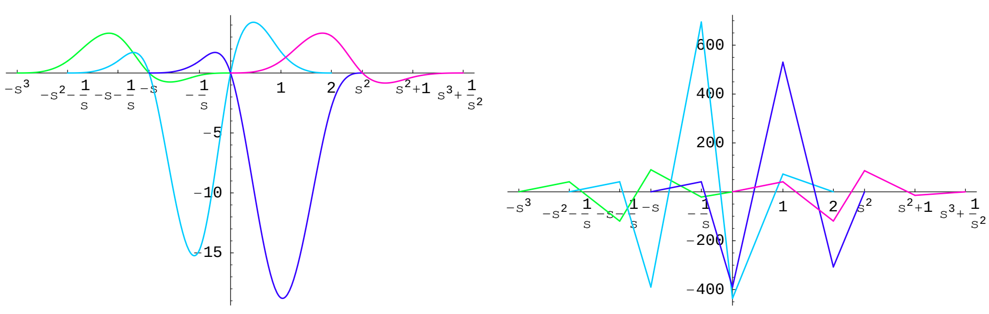

Figure 6: Four cubic functions $\Psi_n^{(4)}(x)$'s and their second derivatives $\zeta_{\lambda_n}(x-\lambda_n)$'s for $n \in \{-5,-4,-2,0\}$.

tile	interval	$\zeta_{\mathrm{LSLLS}}(x-\lambda_{-2})$ k_i	$\zeta_{\mathrm{LSLLS}}(x-\lambda_{-2})$ q_i	$\zeta_{\mathrm{LLSLL}}(x-\lambda_0)$ k_i	$\zeta_{\mathrm{LLSLL}}(x-\lambda_0)$ q_i
L	$[\lambda_{-2}, \lambda_{-1})$	6	0		
S	$[\lambda_{-1}, \lambda_0)$	$-24(1+2\tau)$	6		
L	$[\lambda_0, \lambda_1)$	$3(14+19\tau)$	$-6(3+4\tau)$	6	0
L	$[\lambda_1, \lambda_2)$	$-3(10+19\tau)$	$3(8+11\tau)$	$-\frac{6(-25+42\tau)}{11}$	6
S	$[\lambda_2, \lambda_3)$	$6(4+5\tau)$	$-6(1+4\tau)$	$\frac{36(10+3\tau)}{11}$	$-\frac{36(-6+7\tau)}{11}$
L	$[\lambda_3, \lambda_4)$			$-\frac{42(-1+3\tau)}{11}$	$\frac{36(-1+3\tau)}{11}$
L	$[\lambda_4, \lambda_5)$			$\frac{6(-1+3\tau)}{11}$	$-\frac{6(-1+3\tau)}{11}$
norm		$\sqrt{6(191+309\tau)}\doteq 64.3882$		$\frac{2}{11}\sqrt{6(281+411\tau)}\doteq 13.6981$	

Table 3: Analytical expression for $\zeta_{\mathrm{LSLLS}}(x-\lambda_{-2})$ and $\zeta_{\mathrm{LLSLL}}(x-\lambda_0)$.

$\mu \in \Omega_\psi$ translated to their basic admissible intervals. On each subinterval $[\lambda_i, \lambda_{i+1})$ the function $\zeta_\mu(x-\lambda_\mu)$ is described by a linear function $k_i(x-\lambda_i)+q_i$. The values of k_i's and q_i's are given in these tables.

Remark 2. *It should be noticed from all elements of the extension field $\mathbb{Q}(\tau)$ shown in the tables that imposing $\Psi_n^{(4)}(\lambda_{n+1}) = 1$ is not an optimal choice. The simplest expressions are found with the choice $\Psi_n^{(4)}(\lambda_{n+1}) = 11/3$. The appearing of number 11 is due to an interesting algebraic feature. In order to determine the coefficients of the linear ζ's we have to solve a system of equations with coefficients in the field $\mathbb{Q}(\tau)$, and so all solutions are of the form $\frac{v}{w} \equiv \frac{a+b\tau}{c+d\tau} = \frac{(a+b\tau)(c-d/\tau)}{c^2-d^2+cd} \in \mathbb{Q}(\tau)$. Now, it happens that the "algebraic squared norms" $ww' = c^2 - d^2 + cd$ of denominators are equal to ± 1 (i.e. w is unit in $\mathbb{Z}[\tau]$) or to ± 11. More generally, properties and shapes of wavelets are independent of the fixing $\Psi_n^{(4)}(\lambda_{n+1}) = u$. With the generic latter choice, all values in the tables 2 and 3 should be multiplied by u.*

By dilation and normalization of the $\zeta_\mu(x)$'s we eventually obtain four mother wavelets $\psi_\mu(x) = \frac{\zeta_\mu(\tau^2 x)}{||\zeta_\mu(\tau^2 x)||}$, $\mu \in \Omega_\psi$, see Fig. 7 for their basic admissible translations. Then the set of functions $\{\psi_\mu(x-\kappa_\mu)\}_{\mu\in\Omega_\psi, \kappa_\mu\in\tau^{-2}\Lambda_\mu}$ forms a Riesz basis of $W_0^{(2)}(\Lambda)$ and consequently the set of functions

$$\left\{\tau^j \psi_\mu(\tau^{2j}x - \kappa_\mu)\right\}_{j\in\mathbb{Z},\, \mu\in\Omega_\psi,\, \kappa_\mu\in\tau^{-2}\Lambda_\mu}$$

forms a Riesz basis of $L^2(\mathbb{R})$.

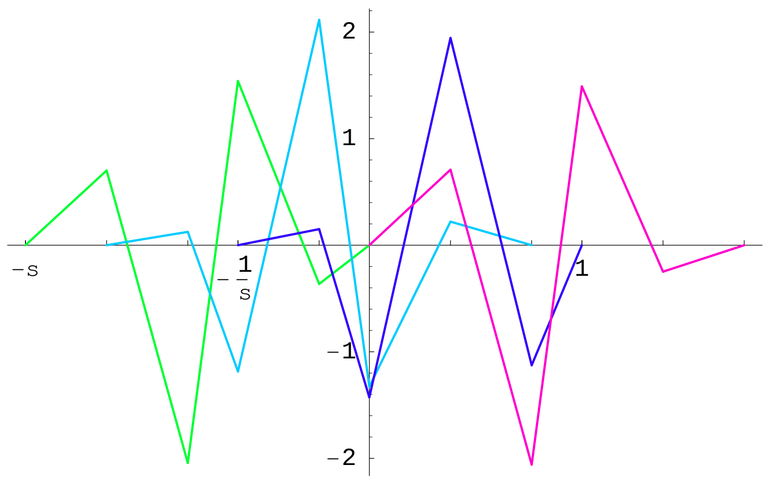

Figure 7: Basic admissible translations of four linear wavelets $\psi_\mu,\ \mu \in \Omega_\psi$.

7.4 Construction of scaling equations for wavelets

Let us here describe the algorithm for obtaining scaling equations in the case of piecewise linear wavelets. For convenience we first set up the equations for the functions $\zeta_\mu,\ \mu \in \Omega_\psi$. The corresponding equations for wavelets $\psi_\mu,\ \mu \in \Omega_\psi$ are trivially derived from them. Let us denote by Ω_ϕ the set of words supporting the scaling functions (in the present case $\Omega_\phi = \{\mathrm{LL}, \mathrm{LS}, \mathrm{SL}\}$). Due to $W_{-1}^{(2)} \subset V_0^{(2)}$ we can write

$$\zeta_\mu(x - \lambda_\mu) = \sum_{\nu \in \Omega_\phi, l \in \Lambda_\nu} g^\mu_{\nu,l} \phi_\nu(x - l).$$

The translate to the point $\lambda_i \in \Lambda_\nu$ of a scaling function ϕ_ν for all $\nu \in \Omega_\phi$ reads as:

$$\phi_{\lambda_i}(x - \lambda_i) = \begin{cases} \dfrac{x - \lambda_i}{\lambda_{i+1} - \lambda_i}, & \text{for } x \in [\lambda_i, \lambda_{i+1}], \\ \dfrac{\lambda_{i+2} - \lambda_i - (x - \lambda_i)}{\lambda_{i+2} - \lambda_{i+1}} = \dfrac{\lambda_{i+2} - x}{\lambda_{i+2} - \lambda_{i+1}}, & \text{for } x \in [\lambda_{i+1}, \lambda_{i+2}]. \end{cases}$$

We look for g_i's in the scaling equation

$$\zeta_{\lambda_i}(x - \lambda_i) = \sum_{j=i}^{i+N_i-2} g_j \phi_{\lambda_j}(x - \lambda_j). \tag{43}$$

- $[\boldsymbol{\lambda_i}, \boldsymbol{\lambda_{i+1}}]$ **:**

Because of support condition and continuity at the point λ_i we can write

$$k_i(x - \lambda_i) + q_i\big|_{x=\lambda_i} = 0,$$

and so the coefficient q_i has to be 0. Hence we have on this interval the equation

$$k_i(x - \lambda_i) = g_i \frac{x - \lambda_i}{\lambda_{i+1} - \lambda_i} \Longrightarrow g_i = k_i(\lambda_{i+1} - \lambda_i).$$

The function is continuous at λ_{i+1} so we have

$$k_i(x - \lambda_i)\big|_{x=\lambda_{i+1}} = k_{i+1}(x - \lambda_{i+1}) + q_{i+1}\big|_{x=\lambda_{i+1}},$$

thus

$$q_{i+1} = k_i(\lambda_{i+1} - \lambda_i) = g_i. \tag{44}$$

- $[\boldsymbol{\lambda_{i+1}}, \boldsymbol{\lambda_{i+2}}]$ **:**

On the interval $[\lambda_{i+1}, \lambda_{i+2}]$ we have the equation

$$k_{i+1}(x - \lambda_{i+1}) + q_{i+1} = g_i \frac{\lambda_{i+2} - x}{\lambda_{i+2} - \lambda_{i+1}} + g_{i+1} \frac{x - \lambda_{i+1}}{\lambda_{i+2} - \lambda_{i+1}}.$$

Using the equation (44) we obtain $g_{i+1} = k_{i+1}(\lambda_{i+2} - \lambda_{i+1}) + q_{i+1}$. From the continuity at the point λ_{i+2} we also have

$$k_{i+1}(x - \lambda_{i+1}) + q_{i+1}\big|_{x=\lambda_{i+2}} = k_{i+2}(x - \lambda_{i+2}) + q_{i+2}\big|_{x=\lambda_{i+2}},$$

thus $q_{i+2} = g_{i+1}$.

- $[\boldsymbol{\lambda_{i+l}}, \boldsymbol{\lambda_{i+l+1}}]$ **:**

By induction we have generally for $l,\ 0 \leq l \leq N_i - 2$

$$g_{i+l} = k_{i+l}(\lambda_{i+l+1} - \lambda_{i+l}) + q_{i+l} = q_{i+l+1}. \tag{45}$$

This method is simple and very general. Actually it can be carried out for any self-similar locally finite Delone set in the case of piecewise linear wavelets.

Finally the scaling equations for basic admissible translations of functions $\psi_\mu,\ \mu \in \Omega_\psi$ have the form

$$\psi_\mu(x - \kappa_\mu) = \sum_{\nu \in \Omega_\phi,\, l \in \Lambda_\nu} g^\mu_{\nu,l} \frac{\phi_\nu(\tau^2 x - l)}{||\zeta_\mu(\tau^2 x)||},\ \kappa_\mu \in \theta^{-1}\Lambda_\mu.$$

	$\psi_{\mathrm{LLSLS}}(x-\kappa_{-5})$	$\psi_{\mathrm{LLSLS}}(x-\kappa_{-4})$	$\psi_{\mathrm{LSLSLL}}(x-\kappa_{-2})$	$\psi_{\mathrm{LLSLL}}(x)$
$\phi_{\mathrm{LL}}(\tau^2 x - \lambda_{-5})$	6			
$\phi_{\mathrm{LS}}(\tau^2 x - \lambda_{-4})$	$\frac{12}{11}(5-13\tau)$	6		
$\phi_{\mathrm{SL}}(\tau^2 x - \lambda_{-3})$	$\frac{12}{11}(4+5\tau)$	$-6(3+4\tau)$		
$\phi_{\mathrm{LS}}(\tau^2 x - \lambda_{-2})$	$\frac{12}{11}(2-3\tau)$	$\frac{6}{11}(58+79\tau)$	6	
$\phi_{\mathrm{SL}}(\tau^2 x - \lambda_{-1})$		$-\frac{18}{11}(13+16\tau)$	$-6(3+4\tau)$	
$\phi_{\mathrm{LL}}(\tau^2 x - \lambda_{0})$		$\frac{3}{11}(13+16\tau)$	$3(49+79\tau)$	6
$\phi_{\mathrm{LS}}(\tau^2 x - \lambda_{1})$			$-6(1+4\tau)$	$\frac{36}{11}(6-7\tau)$
$\phi_{\mathrm{SL}}(\tau^2 x - \lambda_{2})$				$-\frac{36}{11}(1-3\tau)$
$\phi_{\mathrm{LL}}(\tau^2 x - \lambda_{3})$				$\frac{6}{11}(1-3\tau)$

Table 4: Table of coefficients for scaling equations of $\psi_\mu,\ \mu \in \Omega_\psi$.

Coefficients $g^{\mu}_{\nu,l}$ are recalled in Table 4. The norms in denominators are:

$$\begin{aligned}
||\zeta_{\mathrm{LLSLS}}(\tau^2 x)|| &= \frac{2}{11}\sqrt{6(109+161\tau)} \doteq 8.561,\\
||\zeta_{\mathrm{LSLSLL}}(\tau^2 x)|| &= \frac{2}{11}\sqrt{3(6363+10391\tau)} \doteq 47.942,\\
||\zeta_{\mathrm{LSLLS}}(\tau^2 x)|| &= \sqrt{6(73+118\tau)} \doteq 39.794,\\
||\zeta_{\mathrm{LLSLS}}(\tau^2 x)|| &= \frac{2}{11}\sqrt{6(151+130\tau)} \doteq 8.466.
\end{aligned}$$

8 Wavelets for stone-inflation tilings

The construction of spline wavelets for self-similar Delaunay point sets in $\mathbb{R}^d$ has also been considered by Bernuau in [8]. Here we shall briefly describe similar constructions for a tiling $\mathcal{T}$ of $\mathbb{R}^d$ which has the *stone-inflation* symmetry $\sigma\mathcal{T} \subset \mathcal{T}$ where $\sigma = (b, \theta R) \in \mathbb{R}^d \rtimes \left(\mathbb{R}^*_+ \times SO(d)\right)$, $\theta > 1$, is an affine transformation of $\mathbb{R}^d$. Related constructions were also described in [23, 24]. Let us first recall what we understand by tiling. A tiling of $\mathbb{R}^d$ is a covering with non-overlapping pieces all congruent (in the Euclidean group action sense) to tiles belonging to a predetermined finite set of *prototiles* or

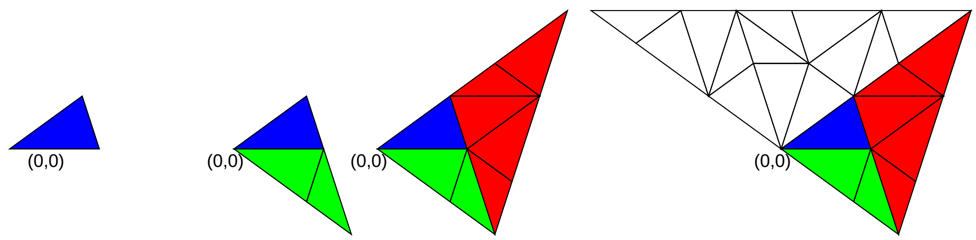

Figure 8: First three steps of stone-inflation for a Penrose tiling.

tile alphabet, $\mathcal{A} = \{T_1, \ldots, T_n\}$:

$$\mathbb{R}^d = \bigcup_{i\in\mathbb{N}} P_i,\ P_i \overset{\text{cong}}{=} T_l,\ T_l \in \mathcal{A},\ P_i \cap P_j \subset \mathbb{R}^{d'},\ i \neq j,\ d' < d. \tag{46}$$

In this definition, a tile is supposed to be compact, equal to the closure of its interior, and homeomorphic to a topological ball. Moreover, in the tiling any two tiles have pairwise disjoint interiors (all tiles of the tiling *pack* $\mathbb{R}^d$). Many tilings $\mathcal{T}$ of $\mathbb{R}^d$ have the so-called stone-inflation symmetry: all tiles of $\mathcal{T}$, when rotated and scaled by a given $\theta R > 1$, $\theta > 1$, $R \in SO(d)$, and translated by a given $b \in \mathbb{R}^d$, can be packed face-to-face from the original ones. An example is provided by a tiling in the Penrose class as shown in Fig. 8. Another example, also relevant to quasicrystalline studies is the three-dimensional Danzer tiling [25].

More precisely, suppose we are given a tiling $\mathcal{T}$ of $\mathbb{R}^d$ built from a finite set of prototiles $\{T_1, \ldots, T_n\}$ present in the tiling,

$$\mathcal{T} = \bigcup_{i\in\mathbb{N}} P_i = \bigcup_{l=1}^{n} \bigcup_{\gamma_l\in\Gamma_l} \gamma_l \cdot T_l, \tag{47}$$

where $\gamma_l = (b_l, R_l)$ is a translation-rotation in the Euclidean group $\mathbb{R}^d \rtimes SO(d)$ and $\Gamma_l \subset \mathbb{R}^d \rtimes SO(d)$ is made of all those transformations $\gamma_l = (b_l, R_l)$ (including the identity) which bring the prototile T_l to one of its congruent companions appearing in the tiling. Here, we keep the freedom to restrict the Γ_l's to pure translation sets, at the price of enlarging the number of prototiles. The stone-inflation symmetry based on the affine-linear inflation $\sigma = (b, \theta R)$ then means that for each l and each $\gamma_l \in \Gamma_l$, the following finite patch

$$\sigma P_i = \sigma\gamma_l \cdot T_l = \bigcup_{m=1}^{n} \bigcup_{\sigma_{lm}\in\Gamma_m} \sigma_{lm} \cdot T_m, \tag{48}$$

is present in the tiling. Hence, it becomes possible to deal with an infinite sequence of inflated-deflated versions of the tiling $\mathcal{T}$

$$\cdots \mathcal{T}_{j-1} \subset \sigma^{-j}\mathcal{T} \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \sigma^{-j} \cdot P_i \stackrel{\text{def}}{=} \mathcal{T}_j \subset \mathcal{T}_{j+1} \cdots . \tag{49}$$

Here, the inclusion relation between two tilings should be understood as: $\mathcal{T} \subset \mathcal{T}'$ if any tile of $\mathcal{T}$ is patch of tiles of $\mathcal{T}'$. Let $\Lambda(\mathcal{T})$ denote the set of vertices of the tiling $\mathcal{T}$. It is clearly Delaunay and we have the counterpart of (49):

$$\cdots \Lambda(\mathcal{T}_{j-1}) \subset \Lambda(\mathcal{T}_j) \subset \Lambda(\mathcal{T}_{j+1}) \cdots . \tag{50}$$

Furthermore, we have the denseness property:

$$\mathbb{R}^d = \overline{\bigcup_j \Lambda(\mathcal{T}_j)}. \tag{51}$$

8.1 Haar wavelet basis

In order to prepare the discussion about the feasibility of finding spline wavelets adapted to tilings of the above type, let us first adapt to our context the construction of an orthonormal Haar wavelet basis of $L^2(\mathbb{R}^d)$ proposed in [8]. We first define a sequence of spaces $(V_j(\mathcal{T}))_{j\in\mathbb{Z}}$ such that each space $V_j(\mathcal{T})$ is the closed subspace of $L^2(\mathbb{R}^d)$ of functions which are constant on all tiles $\sigma^{-j}P_i$, $i \in \mathbb{N}$. Then we have,

Proposition 9. *The sequence* $(V_j(\mathcal{T}))_{j\in\mathbb{Z}}$ *is a* σ-multiresolution analysis *of* $L^2(\mathbb{R}^d)$ *in the following sense:*

(i) *for all* $j \in \mathbb{Z}$, $V_{j-1}(\mathcal{T}) \subset V_j(\mathcal{T})$,

(ii) $\bigcup_{j\in\mathbb{Z}} V_j(\mathcal{T})$ *is dense in* $L^2(\mathbb{R}^d)$,

(iii) $\bigcap_{j\in\mathbb{Z}} V_j(\mathcal{T}) = \{0\}$,

(iv) *for all* $j \in \mathbb{Z}, \mathbf{x} \in \mathbb{R}^d$, $f(\mathbf{x}) \in V_j(\mathcal{T}) \Longleftrightarrow f(\sigma^{-j}\mathbf{x}) \in V_0(\mathcal{T})$,

(v) *there exists a finite number* n *of scaling functions,* $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_n(\mathbf{x})$ *such that all their admissible linear-affine transformations* $\{\phi_l(\gamma_l^{-1} \cdot \mathbf{x})\}_{1\leq l\leq n,\, \gamma_l\in\Gamma_l}$ *form an orthonormal basis in* $V_0(\mathcal{T})$.

Proof.

(i) This inclusion results from the stone-inflation property of tiling $\mathcal{T}$, $\sigma\mathcal{T} \subset \mathcal{T}$.

(ii) This is true through the fact that every continuous function f with compact support on $\mathbb{R}^d$ can be written as uniform limit of the sequence $(f_j)_{j\geq 0}$,

$$f_j(\mathbf{x}) = \sum_{m\in\mathbb{N}} f(\mathbf{x}_{j,m})\mathbf{1}_{\sigma^{-j}P_m}(\mathbf{x}),$$

where $\mathbf{x}_{j,m} \in \sigma^{-j}P_m$.

(iii) By construction it is clear that only the function $f(\mathbf{x}) = 0$ is included in all spaces $V_j(\mathcal{T})$.

(iv) Let us choose $f(\mathbf{x}) \in V_j(\mathcal{T})$. Then we have

$$f(\mathbf{x}) = \sum_{m\in\mathbb{N}} c_m\mathbf{1}_{\sigma^{-j}P_m}(\mathbf{x}), \text{ with } \sum_{m\in\mathbb{N}} |c_m|^2 < \infty,$$

and this is equivalent to

$$f(\sigma^{-j}\mathbf{x}) = \sum_{m\in\mathbb{N}} c_m\mathbf{1}_{P_m}(\mathbf{x}).$$

Thus $f(\sigma^{-j}\mathbf{x}) \in V_0(\mathcal{T})$.

(v) We have n scaling functions $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_n(\mathbf{x})$ which are the normalized characteristic functions of the corresponding tiles $T_l,\ l = 1, \ldots, n$ divided by the square root of its volume, *e.g.* for a tile $X \in \mathcal{A}$

$$\phi_X(\mathbf{x}) = \mathbf{1}_X(\mathbf{x})/\sqrt{|X|}$$

where $|X|$ means the volume of the tile X. Due to normalization we see that all admissible linear-affine transformations of n functions $\{\phi_l\}_{l=1}^n$ form an orthonormal basis of $V_0(\mathcal{T})$.

□

We now proceed to the construction of the Haar wavelets. For each prototile T_l, $1 \leq l \leq n$ there are finitely many tiles P_{l_j}, such that:

$$\sigma T_l = P_{l_1} \cup \cdots \cup P_{l_{k_l}} \iff T_l = \sigma^{-1}P_{l_1} \cup \cdots \cup \sigma^{-1}P_{l_{k_l}},$$

and the set $\left\{P_{l_1}, \ldots, P_{l_{k_l}}\right\}$ forms a patch present in the tiling $\mathcal{T}$. Denote by $V_{0,P_i}(\mathcal{T})$ the subspace of $V_0(\mathcal{T})$ of functions which are zero (almost everywhere) outside the tile P_i. Accordingly we define $V_{j,P_i}(\mathcal{T})$ as the subspace of $V_j(\mathcal{T})$ of functions equal to zero (almost everywhere) outside the tile P_i. Therefore we obtain the orthogonal decompositions:

$$V_0(\mathcal{T}) = \bigoplus_{i\in\mathbb{N}}{}_{\perp} V_{0,P_i}(\mathcal{T}), \tag{52}$$

$$V_j(\mathcal{T}) = \bigoplus_{i\in\mathbb{N}}{}_{\perp} V_{j,P_i}(\mathcal{T}), \tag{53}$$

and the inclusions

$$V_{0,P_i}(\mathcal{T}) \subset V_{1,P_i}(\mathcal{T}) \subset \cdots \subset V_{j,P_i}(\mathcal{T}) \cdots . \tag{54}$$

The wavelet space $W_0(\mathcal{T})$ is the orthogonal complement of $V_0(\mathcal{T})$ in $V_1(\mathcal{T})$,

$$V_1(\mathcal{T}) = V_0(\mathcal{T}) \oplus_{\perp} W_0(\mathcal{T}). \tag{55}$$

More generally

$$V_{j+1}(\mathcal{T}) = V_j(\mathcal{T}) \oplus_{\perp} W_j(\mathcal{T}).$$

We also define the orthogonal complement of $V_{j,P_i}(\mathcal{T})$ in $V_{j+1,P_i}(\mathcal{T})$ as

$$V_{j+1,P_i}(\mathcal{T}) = V_{j,P_i}(\mathcal{T}) \oplus_{\perp} W_{j,P_i}(\mathcal{T}). \tag{56}$$

It is clear that

$$W_j(\mathcal{T}) = \bigoplus_{i\in\mathbb{Z}}{}_{\perp} W_{j,P_i}(\mathcal{T}).$$

Finally we get the orthogonal decomposition of the whole Hilbert space:

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}}{}_{\perp} W_j(\mathcal{T}).$$

Thus the construction of $W_0(\mathcal{T})$ is equivalent to the construction of all $W_{0,P_i}(\mathcal{T})$'s. Since any tile P_i can be written as

$$P_i = \gamma_l T_l = \gamma_l(\sigma^{-1}P_{l_1} \cup \cdots \cup \sigma^{-1}P_{l_{k_l}}),$$

it is sufficient to find wavelets for prototiles $T_l,\ l = 1, \ldots, n$, and the whole basis of $W_0(\mathcal{T})$ will be formed by all admissible linear-affine transformations of these "protowavelets". There results the following proposition.

Proposition 10. *For every prototile T_l, $l = 1, \dots, n$, given also by*

$$T_l = \sigma^{-1} P_{l_1} \cup \dots \cup \sigma^{-1} P_{l_{k_l}},$$

we have $k_l - 1$ orthonormal wavelets

$$\psi_{1,l}(\mathbf{x}), \dots, \psi_{k_l-1,l}(\mathbf{x}).$$

Proof. Let us denote by $V_{0,T_l}(\mathcal{T})$ the space of functions constant on tile T_l and equal to zero otherwise, *i.e.* those functions proportional to $\mathbf{1}_{T_l}(\mathbf{x})$. We then denote by $V_{1,T_l}(\mathcal{T})$ the space of functions constant on tiles $\sigma^{-1}P_{l_1}, \dots, \sigma^{-1}P_{l_{k_l}}$ and otherwise equal to zero. The space of wavelets corresponding to the tile T_l is found as the orthogonal complement of $V_{0,T_l}(\mathcal{T})$ in $V_{1,T_l}(\mathcal{T})$, i.e.

$$V_{1,T_l}(\mathcal{T}) = V_{0,T_l}(\mathcal{T}) \oplus_{\perp} W_{0,T_l}(\mathcal{T}).$$

One possible construction of an orthonormal basis of $W_{0,T_l}(\mathcal{T})$ is as follows: we start from the set of functions $\{\mathbf{1}_{T_l}(\mathbf{x}), \mathbf{1}_{\sigma^{-1}P_{l_1}}(\mathbf{x}), \dots, \mathbf{1}_{\sigma^{-1}P_{l_{k_l-1}}}(\mathbf{x})\}$ which form a basis of $V_{1,T_l}(\mathcal{T})$. Because of the presence of the function $\mathbf{1}_{T_l}(\mathbf{x})$ we have put aside (for instance) the last function $\mathbf{1}_{\sigma^{-1}P_{l_{k_l}}}(\mathbf{x})$. We then proceed to Gram-Schmidt orthogonalization and L^2-normalization. Thus we get an orthonormal basis of $V_{1,T_l}(\mathcal{T})$ and after removing the single basis element in $V_{0,T_l}(\mathcal{T})$ we are left with $k_l - 1$ functions which form an orthonormal basis of $W_{0,T_l}(\mathcal{T})$. $\square$

The orthonormal Haar basis of $L^2(\mathbb{R}^d)$ adapted to the tiling $\mathcal{T}$ is finally given by:

$$\bigcup_{j\in\mathbb{Z}} \bigcup_{1\leq l\leq n} \bigcup_{\gamma_l\in\Gamma_l} \left\{\theta^{j\frac{d}{2}}\psi_{l_1,l}\left(\gamma_l^{-1}\sigma^j\cdot\mathbf{x}\right), \dots, \theta^{j\frac{d}{2}}\psi_{l_{k_l-1},l}\left(\gamma_l^{-1}\sigma^j\cdot\mathbf{x}\right)\right\}.$$

8.2 Beyond Haar: spline wavelet basis

Let $\mathcal{T}$ be a stone inflation tiling of $\mathbb{R}^d$. Suppose prototiles are polytopes and the set $\Lambda(\mathcal{T})$ of vertices is of finite local complexity. How to build piecewise linear, compactly supported, spline scaling functions and related wavelets living on tiles of $\mathcal{T}$? More precisely, one could think about going through the following steps.

- How many tiles are needed for building the minimal patches supporting scaling functions?

- How many different patches of such type exist in the tiling?
- How to characterize such spline functions from a functional analysis point of view?
- Suppose the previous questions answered. How to build the corresponding multiresolution?
- Suppose the previous question be solved and related scaling functions be determined. How to find the corresponding wavelets?
- Are these wavelets compactly supported?
- If yes, on which patches in the tiling?
- How to extend this material to smoother splines?

It is not the aim of the present paper to achieve this ambitious program. Let us just sketch which procedure could be followed in the simplest two-dimensional case. In order to build "pyramidal" functions (the two-dimensional analogue of the one dimensional spline "hat" functions) which are to play the role of scaling functions, we have to accomplish a triangulation of the tiling $\mathcal{T}$ (like we already have for Penrose tilings). By triangulation we mean that in each prototile some of the vertices could be pairwise connected with a new segment (in order to divide the prototile into triangles) under the condition not to create new nodes (the new edges should not cross). All tiles congruent to the divided prototile will be triangulated in the same manner. Then to every point $\lambda \in \Lambda(\mathcal{T})$ there is an associated pyramid. The support of this pyramid is delimited by the neighbours of λ. (A neighbour is a point connected with λ by an edge). The vertex of the pyramid is obviously located right above λ. Because of the finite local complexity we know that there exists a finite number of different pyramids. These pyramidal functions are uniquely determined once a certain normalization condition is fulfilled. These pyramids and all their admissible linear-affine transformation then define the space $V_0(\mathcal{T})$. The multiresolution analysis follows.

A simple example is provided by the square lattice $\Lambda = \mathbb{Z}^2$: we just have to divide each square in the same manner into two triangles. Then we get one scaling function which is a "hexagonal pyramid", as is shown in Fig. 9.

Another example is provided by a five-fold Penrose tiling. Seven pyramidal functions exist here. As is shown in Fig. 10, their respective supports

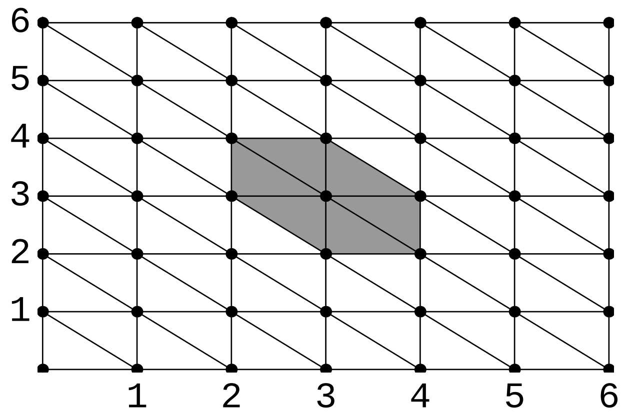

Figure 9: Support of "hexagonal" pyramidal function for the square lattice $\Lambda = \mathbb{Z}^2$.

are (in terms of kites, darts and rhombuses): a decagonal patch of 5 kites, a star-shaped patch of 5 darts, a kite-shaped patch of 2 kites and 1 "fat" rhombus (*i.e* 2 obtuse triangles), a rhombus-shaped patch of 1 dart and 1 kite (an "ace"), a patch of 3 darts and 1 "thin" rhombus (*i.e* 2 acute triangles), a patch of 1 dart and 2 thin rhombuses, and a patch of 3 kites and 2 obtuse triangles.

9 Conclusion

The Bernuau construction of wavelets adapted to self-similar aperiodic point sets has been carried out in dealing with some one-dimensional examples. Our aim is currently to apply these wavelets to the analysis of aperiodic structures, like diffraction spectra of Fibonacci chain, and to compare our results with more standard wavelet analysis (*e.g.* dyadic wavelets). Of course, an essential step in decomposition and recomposition schemes will be the determination of corresponding biorthogonal basis. We also plan to extend our constructions to higher-dimensional cases, like Penrose or Danzer tilings, in view of practical applications to quasicrystalline studies.

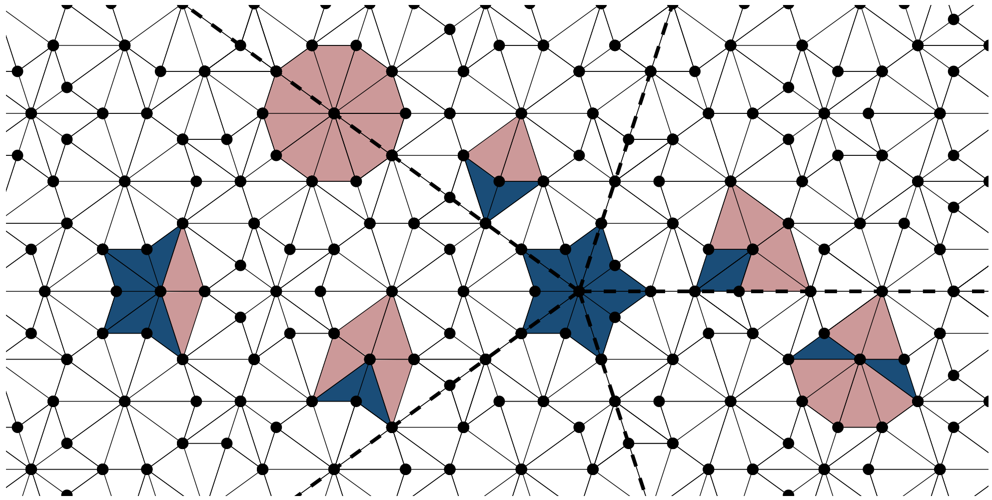

Figure 10: Supports of seven different pyramidal functions for a (five-fold symmetrical) Penrose tiling.

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